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Previously Unknown Formulas for the Relativistic Kinetic Energy of an Electron in a Hydrogen Atom

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Abstract

Einstein's energy-momentum relationship, which holds in an isolated system in free space, contains two formulas for relativistic kinetic energy. Einstein's relationship is not applicable in a hydrogen atom, where potential energy is present. However, a relationship similar to that can be derived. That derived relationship also contains two formulas, for the relativistic kinetic energy of an electron in a hydrogen atom. Furthermore, it is possible to derive a third formula for the relativistic kinetic energy of an electron from that relationship. Next, the paper looks at the fact that the electron has a wave nature. Five more formulas can be derived based on considerations relating to the phase velocity and group velocity of the electron. This paper presents eight formulas for the relativistic kinetic energy of an electron in a hydrogen atom.

Keywords

Einstein's Energy-Momentum Relationship, Relativistic Kinetic Energy, Bohr's Quantum Condition, Potential Energy, Phase Velocity, Group Velocity

1. Introduction

In classical mechanics, the kinetic energy K_{cl} of a body with mass m is given by the following formula.

$$K_{\mathrm{cl}}=\frac{1}{2}mv^2. \tag{1}$$

Here, the subscript "cl" of K indicates that this is the formula for classical mechanics.

Classical mechanics does not take into account the special theory of relativity

(STR), and thus there is no need to distinguish between rest mass and relativistic mass.

However, in Einstein's STR, the two types of mass must be distinguished.

According to the STR, the following relation holds between the energy and momentum of a body moving in free space [1].

$$\left(mc^2\right)^2 = \left(m_0c^2\right)^2 + c^2p^2. \tag{2}$$

Here, m_0c^2 is the rest mass energy of the body. And mc^2 is the relativistic energy.

In the STR, there is the following relationship between rest mass m_0 and relativistic mass m.

$$m = m_0\left(1-\beta^2\right)^{-1/2}, \quad \beta = v/c. \tag{3}$$

When β is extremely small, Equation (3) can be expanded as a power series in β, as indicated below.

$$m = m_0\left(1-\beta^2\right)^{-1/2} = m_0\left(1+\frac{1}{2}\beta^2+\frac{3}{8}\beta^4+\cdots\right). \tag{4}$$

It is known that, if a body is moving at low velocity, Equation (4) can be approximated at high precision using the first two terms. Thus,

$$mc^2 \approx m_0c^2\left(1+\frac{1}{2}\beta^2\right) = m_0c^2+\frac{1}{2}m_0v^2. \tag{5}$$

The second term on the right side of Equation (5) is the kinetic energy in Newtonian mechanics.

Now, how is the relativistic kinetic energy of a body defined in the STR?

Einstein and Sommerfeld defined the relativistic kinetic energy K_{re} as follows [2].

$$K_{\text{re}} = mc^2 - m_0c^2. \tag{6}$$

The "re" subscript of K_{re} stands for "relativistic."

If the relativistic kinetic energy of a body is defined with Equation (6), then it is possible to derive yet another formula for K_{re}. That formula is found as follows [3].

Now, Equation (2) is rewritten as follows.

$$\left(mc^2\right)^2 = m_0^2c^4 + \left(m^2c^4 - m_0^2c^4\right) = \left(m_0c^2\right)^2 + c^2p^2. \tag{7}$$

Comparing Equations (6) and (7), the relativistic momentum p_{re} can be defined as follows.

$$p_{\text{re}}^2 = m^2c^2 - m_0^2c^2. \tag{8}$$

Hence,

$$p_{\text{re}}^2 = \left(m+m_0\right)\left(mc^2 - m_0c^2\right). \tag{9}$$

The following relation holds due to Equations (6) and (9).

$$K_{\text{re}} = \frac{p_{\text{re}}^2}{m + m_0}. \tag{10}$$

Based on the above discussion, it was found that the relativistic kinetic energy of a body moving in isolated systems in free space can be described with Equations (6) and (10).

2. The Relativistic Kinetic Energy of an Electron in a Hydrogen Atom, Part 1

The author has previously derived a number of formulas for the relativistic kinetic energy of an electron in a hydrogen atom. This paper brings all of those formulas together. First, Section 2 derives a formula for the relativistic kinetic energy of an electron in a hydrogen atom by referring to Equations (6) and (10).

An energy-momentum relationship applicable to the electron in a hydrogen atom has already been derived in a previous paper [4] [5].

Naturally, this relationship should be derived mathematically. However, we can also predict this relationship based on simple considerations.

Now, consider the case where an electron at rest in an isolated system in free space is attracted by the electrostatic attraction of the proton (hydrogen atom nucleus), and forms a hydrogen atom.

The electron at rest has a rest mass energy of $m_{\text{e}}c^2$. When this electron is taken into the region of the hydrogen atom, it acquires an amount of kinetic energy K_{re} equivalent to the emitted photon.

Now, the following relationship holds if the energy of a photon emitted from an electron is taken to be $h\nu$.

$$h\nu = K_{\text{re}}. \tag{11}$$

If an electron at rest in free space acquires kinetic energy by emitting a photon, then an energy source is needed for that energy. Normally, we believe that the energies $h\nu$ and K_{re} are supplied by the electron reducing its potential energy. However, although potential energy has a name, it has no real substance. The only energy of an electron at rest in an isolated system in free space is rest mass energy. Thus, the author had the idea that the reduction in rest mass energy of the electron corresponds to potential energy [6] [7]. Here, if we represent the reduction in rest mass energy of the electron as $-\Delta m_{\text{e}}c^2$, then the potential energy can be defined as follows.

$$V(r) = -\Delta m_{\text{e}}c^2. \tag{12}$$

Also, if the law of energy conservation is taken into account, then the following relationship holds.

$$-\Delta m_{\text{e}}c^2 + h\nu + K_{\text{re}} = 0. \tag{13}$$

Next, the relativistic energy of an electron in a hydrogen atom, $m_n c^2$, is defined as follows.

$$m_n c^2 = m_{\text{e}}c^2 - h\nu = m_{\text{e}}c^2 + V(r_n) + K_{\text{re},n}, \quad n = 1, 2, \cdots. \tag{14}$$

Here, n is the principal quantum number.

The relativistic energy of an electron in a hydrogen atom becomes smaller than the rest mass energy. That is,

$$m_n c^2 < m_e c^2. \tag{15}$$

The behavior of an electron inside an atom, where there is potential energy, cannot be described with the relationship of Einstein Equation (2).

Now, referring to Equation (6), it is natural to define the relativistic kinetic energy of an electron in a hydrogen atom as follows [8].

$$K_{\mathrm{re},n} = -E_{\mathrm{re},n} = m_e c^2 - m_n c^2. \tag{16}$$

Also, this paper defines $E_{\mathrm{re},n}$ as the relativistic energy levels of the hydrogen atom.

However, the term "relativistic" used here does not mean based on the STR. It means that the expression takes into account the fact that the mass of the electron varies due to velocity. According to the STR, the electron's mass increases when its velocity increases. However, inside the hydrogen atom, the mass of the electron decreases when the velocity of the electron increases.

Next, the relativistic kinetic energy of an electron in a hydrogen atom is defined as follows by referring to Equation (10).

$$K_{\mathrm{re},n} = \frac{p_{\mathrm{re},n}^2}{m_e + m_n}, \quad p_{\mathrm{re},n} = m_n v_n. \tag{17}$$

In this way, two formulas have been obtained for the relativistic kinetic energy of the electron in a hydrogen atom (Equations (16), and (17)).

The following equation can be derived from Equations (16) and (17).

$$m_e c^2 - m_n c^2 = \frac{p_{\mathrm{re},n}^2}{m_e + m_n}. \tag{18}$$

Rearranging this, the following relationship can be derived.

$$\left(m_n c^2\right)^2 + c^2 p_{\mathrm{re},n}^2 = \left(m_e c^2\right)^2. \tag{19}$$

Equation (19) is the energy-momentum relationship applicable to the electron in a hydrogen atom.

In the past, Dirac derived the following negative solution from Equation (2).

$$E = \pm mc^2 = \pm m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{20}$$

If the same logic is applied to Equation (19), then the following formula can be derived.

$$E = \pm m_n c^2 = \pm m_e c^2 \left(1 + \frac{v_n^2}{c^2}\right)^{-1/2}. \tag{21}$$

However, Equation (21) does not incorporate the discontinuity peculiar to the micro world. Therefore, Equation (21) must be rewritten into a relationship where energy is discontinuous.

The author has previously derived the following relationship as a new quantum condition to replace the quantum condition of Bohr [9].

$$\frac{v_n}{c} = \frac{\alpha}{n}. \tag{22}$$

Here, α is the following fine-structure constant.

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}. \tag{23}$$

Using the relation in Equation (22), Equation (21) can be written as follows.

$$\pm m_n c^2 = \pm m_{\mathrm{e}} c^2 \left(1 + \frac{\alpha^2}{n^2}\right)^{-1/2} \tag{24a}$$

$$= \pm m_{\mathrm{e}} c^2 \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}. \tag{24b}$$

Hence,

$$\frac{m_n}{m_{\mathrm{e}}} = \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}. \tag{25}$$

The relativistic kinetic energy of the electron can be expressed as follows.

$$K_{\mathrm{re},n} = -E_{\mathrm{re},n} = m_{\mathrm{e}} c^2 - m_n c^2 \tag{26a}$$

$$= m_{\mathrm{e}} c^2 \left[1 - \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}\right]. \tag{26b}$$

3. A New Quantum Condition More Useful Than Bohr's Quantum Condition, Equation (22)

It was possible to derive Bohr's quantum condition from Equation (22).

The energy levels of a hydrogen atom derived by Bohr, and its relativistic energy levels, can also be derived from Equation (22).

In 1913, Bohr derived the following formulas for the energy levels of a hydrogen atom, and the orbital radius of the electron orbiting inside the atom [10].

$$E_{\mathrm{BO},n} = -\frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{m_{\mathrm{e}} e^4}{\hbar^2} \cdot \frac{1}{n^2} = -\frac{\alpha^2 m_{\mathrm{e}} c^2}{2n^2}, \quad n = 1, 2, \cdots. \tag{27}$$

$$r_{\mathrm{BO},n} = 4\pi\varepsilon_0 \frac{\hbar^2}{m_{\mathrm{e}} e^2} \cdot n^2, \quad n = 1, 2, \cdots. \tag{28}$$

The subscript "BO" signifies a physical quantity predicted by Bohr.

When deriving Equations (27) and (28), Bohr assumed the following quantum condition.

$$m_{\mathrm{e}} v_n \cdot 2\pi r_{\mathrm{BO},n} = 2\pi n \hbar. \tag{29}$$

First, both sides of Equation (22) are multiplied by $m_{\mathrm{e}} \cdot 2\pi r_n$.

$$m_e v_n \cdot 2\pi r_n = \frac{m_e c\alpha}{n} \cdot 2\pi r_n. \tag{30}$$

Next, if Equation (23) is substituted for α and Equation (28) is substituted for r_n on the right side of Equation (30), we obtain the following.

$$m_e v_n \cdot 2\pi r_n = \frac{m_e c}{n}\left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right) 2\pi \cdot 4\pi\varepsilon_0 \frac{\hbar^2}{m_e e^2} n^2 = 2\pi n\hbar. \tag{31}$$

With this, it was possible to derive Bohr's quantum condition (29) from Equation (22).

Next, Equation (27) is derived from Equation (22).

When both sides of Equation (22) are squared, and then multiplied by $m_e/2$,

$$\frac{1}{2}\frac{m_e v_n^2}{c^2} = \frac{1}{2}\frac{m_e \alpha^2}{n^2}. \tag{32}$$

Hence,

$$E_n = -\frac{1}{2} m_e v_n^2 = -\frac{\alpha^2 m_e c^2}{2n^2}. \tag{33}$$

If Equation (22) is taken as a departure point, the energy levels of the hydrogen atom derived by Bohr can be derived immediately. However, from a relativistic perspective, $(1/2) m_e v_n^2$ is an approximation of the kinetic energy of the electron. Therefore, the energy in Equation (33) is also an approximation of the true value.

Next, the relativistic energy levels of a hydrogen atom are derived from Equation (22).

First, if both sides of Equation (22) are squared, and multiplied by $m_n^2/(m_e + m_n)$,

$$\frac{m_n^2}{m_e + m_n} \cdot \frac{v_n^2}{c^2} = \frac{\alpha^2}{n^2} \cdot \frac{m_n^2}{m_e + m_n}. \tag{34}$$

From this, the relativistic energy of the hydrogen atom $E_{\text{re},n}$ is,

$$E_{\text{re},n} = -\frac{m_n^2 v_n^2}{m_e + m_n} = -\frac{\alpha^2 c^2}{n^2} \cdot \frac{m_n^2}{m_e + m_n}. \tag{35}$$

If the relationship in Equation (24) is used here,

$$E_{\text{re},n} = -\frac{\alpha^2 c^2}{n^2}\left(\frac{n^2}{n^2+\alpha^2}\right) m_e^2 \cdot \frac{1}{m_e\left[1+\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}\right]}. \tag{36}$$

Next, the following formula is multiplied with the numerator and denominator,

$$1-\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}.$$

When this is done,

$$E_{\mathrm{re},n} = -\frac{\alpha^2 m_{\mathrm{e}} c^2}{n^2}\left(\frac{n^2}{n^2+\alpha^2}\right)\left[1-\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}\right]\left(1-\frac{n^2}{n^2+\alpha^2}\right)^{-1} \tag{37a}$$

$$= \frac{\alpha^2 m_{\mathrm{e}} c^2}{n^2}\left(\frac{n^2}{n^2+\alpha^2}\right)\left[\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}-1\right]\left(\frac{n^2+\alpha^2}{\alpha^2}\right) \tag{37b}$$

$$= m_{\mathrm{e}} c^2\left[\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}-1\right] \tag{37c}$$

$$= m_{\mathrm{e}} c^2\left[\left(1+\frac{\alpha^2}{n^2}\right)^{-1/2}-1\right]. \tag{37d}$$

This enables derivation of Equation (26b) from Equation (22).

When the part of Equation (37d) in parentheses is expressed as a Taylor expansion,

$$K_{\mathrm{re},n} = -E_{\mathrm{re},n} \approx m_{\mathrm{e}} c^2\left[1-\left(1-\frac{\alpha^2}{2n^2}+\frac{3\alpha^4}{8n^4}-\frac{5\alpha^6}{16n^6}\right)\right] \approx \frac{\alpha^2 m_{\mathrm{e}} c^2}{2n^2}. \tag{38}$$

Here, based on Equation (22),

$$\alpha^2 = \frac{v_n^2 n^2}{c^2}. \tag{39}$$

If α^2 in Equation (39) is substituted for Equation (38),

$$E_{\mathrm{re},n} \approx -\frac{m_{\mathrm{e}} c^2}{2n^2}\cdot\frac{v_n^2 n^2}{c^2} = -\frac{1}{2} m_{\mathrm{e}} v_n^2 = E_{\mathrm{BO},n}. \tag{40}$$

This shows that Equation (27) for the energy levels of the hydrogen atom derived by Bohr is an approximation of Equation (37d).

Only $E_{\mathrm{BO},n}$ can be derived from Bohr's quantum condition Equation (29). In contrast, $E_{\mathrm{BO},n}$ and $E_{\mathrm{re},n}$ can be derived from the new quantum condition Equation (22). Equation (22) is superior to Equation (29).

Incidentally, in Equation (27) for the energy levels of the hydrogen atom derived by Bohr, the energy of an electron at rest infinitely far from the proton was regarded as zero (**Figure 1**).

The rest mass energy of the electron is not taken into account in Bohr's theory. Thus, the author derived a Equation (37) for the energy levels of the hydrogen atom, taking into account the rest mass energy of the electron [11] (**Figure 2**).

4. *r* Corresponding to the Relativistic Energy Levels of an Electron in a Hydrogen Atom

There are still other formulas besides Equation (26) for the relativistic kinetic energy of an electron in a hydrogen atom. Those still unknown formulas will be derived in Section 6, and Sections 4 and 5 are preparation leading up to that.

Now, the total mechanical energy of the hydrogen atom is given by the following formula.

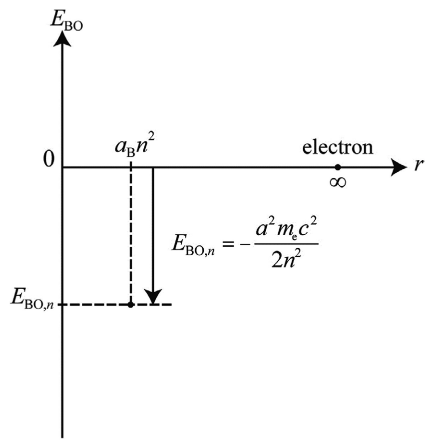

Figure 1. In Bohr's theory, the energy when the electron is at rest at a position infinitely distant from the proton (atomic nucleus) is defined to be zero.

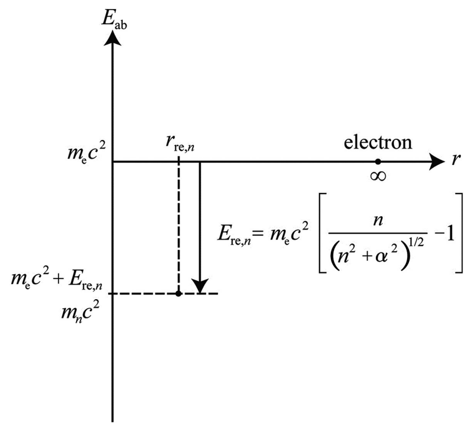

Figure 2. According to the STR, the energy of an electron at rest at a position where $r=\infty$ is $m_e c^2$. $E_{re,n}$ is given by the difference between $m_e c^2$ and $m_n c^2$. That is, $m_n c^2 - E_{re,n} = m_e c^2$.

$$E_{re,n} = K_{re,n} + V(r_n) = -K_{re,n}. \tag{41}$$

Also, if the formula for potential energy is used, then $E_{re,n}$ can be written as follows.

$$E_{re,n} = \frac{1}{2}V(r_n) = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_n} = -\frac{1}{2}m_e c^2\frac{r_e}{r_n} = -m_e c^2\left(\frac{r_e/2}{r_n}\right). \tag{42}$$

From Equation (42), $m_n c^2$ is:

$$m_n c^2 = m_e c^2 + E_{re,n} = m_e c^2 - m_e c^2\left(\frac{r_e/2}{r_n}\right) = m_e c^2\left(\frac{r_n - r_e/2}{r_n}\right). \tag{43}$$

The following equation holds due to Equations (24b) and (43).

$$\frac{n^2}{n^2+\alpha^2} = \left(\frac{r_n - r_e/2}{r_n}\right)^2. \tag{44}$$

From this, the following quadratic equation is obtained.

$$r_n^2 - \left(\frac{n^2+\alpha^2}{\alpha^2}\right) r_e r_n + \left(\frac{n^2+\alpha^2}{\alpha^2}\right)\frac{r_e^2}{4} = 0. \quad (45)$$

If this equation is solved for r_n,

$$r_n^{\pm} = \frac{r_e}{2}\left(1+\frac{n^2}{\alpha^2}\right)\left[1 \pm \left(1+\frac{\alpha^2}{n^2}\right)^{-1/2}\right]. \quad (46)$$

Next, if the electron orbital radii corresponding to the energy levels in Equation (21) are taken to be, respectively, $r_{re,n}^{+}$ and $r_{re,n}^{-}$,

$$r_{re,n}^{+} = \frac{r_e}{2}\frac{\left(n^2+\alpha^2\right)^{1/2}}{\left(n^2+\alpha^2\right)^{1/2}-n}. \quad (47)$$

$$r_{re,n}^{-} = \frac{r_e}{2}\frac{\left(n^2+\alpha^2\right)^{1/2}}{\left(n^2+\alpha^2\right)^{1/2}+n}. \quad (48)$$

Also, Equations (47) and (48) can be written as follows [12].

$$r_{re,n}^{+} = \frac{r_e}{2}\left[1+\frac{n}{\left(n^2+\alpha^2\right)^{1/2}-n}\right]. \quad (49)$$

$$r_{re,n}^{-} = \frac{r_e}{2}\left[1-\frac{n}{\left(n^2+\alpha^2\right)^{1/2}+n}\right]. \quad (50)$$

Here, the subscript "re" is attached to r for consistency with $E_{re,n}$.

The next compares the orbital radii of an electron in a hydrogen atom $r_{re,n}^{+}$ and the orbital radii of an electron with a negative mass $r_{re,n}^{-}$.

The following ratio is obtained from Equations (47) and (48).

$$\frac{r_{re,n}^{-}}{r_{re,n}^{+}} = \frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left(n^2+\alpha^2\right)^{1/2}+n}. \quad (51)$$

In this paper, $r_{re,n}^{+}$ is called the orbital radius, as is customary. However, a picture of the motion of the electron cannot be drawn, even if that motion is discussed at the level of classical quantum theory. The electron in a hydrogen atom is not in orbital motion around the atomic nucleus. The domain of the ordinary hydrogen atom that we all know starts from $r = r_e/2\left(m_n c^2 = 0\right)$.

With the aid of quantum mechanics, $r_{re,n}^{+}$ and $r_{re,n}^{-}$ can be regarded as the locations (positions) where presence of the electron has maximum probability.

The negative solutions for E and r have been discussed in another paper [5]. Therefore, that problem is not considered in this paper.

Incidentally, the following equation holds due to Equations (42) and (26a).

$$\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_{re,n}} = m_e c^2 - m_n c^2. \quad (52)$$

Finding $r_{\mathrm{re},n}$ from Equation (52),

$$r_{\mathrm{re},n} = \frac{r_{\mathrm{e}}}{2}\frac{m_{\mathrm{e}}}{m_{\mathrm{e}} - m_n}. \tag{53}$$

Next, if we calculate the denominator of Equation (53),

$$\frac{m_{\mathrm{e}}}{m_{\mathrm{e}} - m_n} = \frac{1}{1 - \dfrac{1}{\left(1 - \alpha^2/n^2\right)^{1/2}}} = \frac{\left(n^2 + \alpha^2\right)^{1/2}}{\left(n^2 + \alpha^2\right)^{1/2} - n}. \tag{54}$$

Also, the following relationship holds due to Equations (25) and (44).

$$\frac{m_n}{m_{\mathrm{e}}} = \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2} = \frac{r_{\mathrm{re},n}^{+} - r_{\mathrm{e}}/2}{r_{\mathrm{re},n}^{+}}. \tag{55}$$

Here, we rewrite Equation (19) using previously obtained formulas. First, Equation (22) has the following physical meaning.

$$\frac{v_n}{c} = \frac{m_n v_n}{m_n c} = \frac{p_{\mathrm{re},n}}{m_n c} = \frac{\alpha}{n}. \tag{56}$$

Next, Equation (25) can be written as follows.

$$\frac{m_n c}{m_{\mathrm{e}} c} = \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}. \tag{57}$$

Thus,

$$\begin{aligned} \frac{p_{\mathrm{re},n}}{m_{\mathrm{e}} c} &= \frac{p_{\mathrm{re},n}}{m_n c} \cdot \frac{m_n c}{m_{\mathrm{e}} c} \\ &= \frac{\alpha}{n}\left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2} \\ &= \left(\frac{\alpha^2}{n^2 + \alpha^2}\right)^{1/2}. \end{aligned} \tag{58}$$

Using Equation (58), $cp_{\mathrm{re},n}$ is,

$$cp_{\mathrm{re},n} = c \cdot m_{\mathrm{e}} c \left(\frac{\alpha^2}{n^2 + \alpha^2}\right)^{1/2}. \tag{59}$$

Finally, Equation (19) can be written as follows using Equations (25) and (59).

$$\left[m_{\mathrm{e}} c^2 \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}\right]^2 + c^2 \left[m_{\mathrm{e}} c \left(\frac{\alpha^2}{n^2 + \alpha^2}\right)^{1/2}\right]^2 = \left(m_{\mathrm{e}} c^2\right)^2. \tag{60}$$

5. The Relationship between Phase Velocity and Group Velocity of an Electron Wave in a Hydrogen Atom

In parts of Sections 5 and Section 6, the points needed for discussion in this paper are quoted from Ref. [13].

According to Maxwell's electromagnetism, the following relationship holds between the momentum p and energy E of light.

$$E = cp. \tag{61}$$

If a photon as a single particle is assumed to have a frequency ν, Einstein concluded it has the following energy.

$$E = h\nu. \tag{62}$$

Here, h is the Planck constant. Also Equation (62) can be written as follows using the angular frequency ω.

$$E = \hbar\omega, \ \hbar = \frac{h}{2\pi}. \tag{63}$$

ω is defined as follows.

$$\omega = 2\pi\nu. \tag{64}$$

The following equation can be derived from Equations (61) and (62).

$$\lambda = \frac{c}{\nu} = \frac{h}{p}. \tag{65}$$

Also, the wavenumber κ is defined as follows.

$$\kappa = \frac{\lambda}{2\pi}. \tag{66}$$

de Broglie applied Equation (65) to matter, in classical physics, the following relation holds between momentum p and kinetic energy K.

$$K_{\text{cl}} = \frac{1}{2}mv^2 = \frac{p^2}{2m}. \tag{67}$$

Here, if Equations (63), (65), and (66) are used,

$$\hbar\omega = \frac{p^2}{2m} = \frac{1}{2m}\frac{h^2}{\lambda^2} = \frac{1}{2m}\frac{4\pi^2\hbar^2}{\lambda^2} = \frac{\kappa^2\hbar^2}{2m}. \tag{68}$$

Therefore,

$$\omega = \frac{\hbar k^2}{2m}. \tag{69}$$

The phase velocity v_{phase} and group velocity v_{group} of a material wave are defined as follows (in the following, these may be abbreviated as $v_{\text{p}}, v_{\text{g}}$.)

$$v_{\text{phase}} = \frac{\omega}{k}, \ v_{\text{group}} = \frac{\text{d}\omega}{\text{d}k}. \tag{70}$$

In light of the above, the phase velocity of the wave is as follows.

$$v_{\text{p}} = \frac{\omega}{k} = \frac{\hbar k}{2m} = \frac{p}{2m} = \frac{v}{2}. \tag{71}$$

Also, the group velocity of the wave is as follows.

$$v_{\text{g}} = \frac{\text{d}\omega}{\text{d}k} = \frac{\hbar 2k}{2m} = \frac{p}{m} = v. \tag{72}$$

6. The Relativistic Kinetic Energy of an Electron in a Hydrogen Atom, Part 2

The electron's phase velocity $v_{\text{p},n}$ is given by the following formula.

$$v_{p,n} = \lambda_n \nu_n. \tag{73}$$

Here, $v_{p,n}$ is the phase velocity of the electron wave when the principal quantum number is in the n state. Also, λ_n and ν_n are the wavelength and frequency of the electron wave.

Equation (73) can be written as follows using the relationship of Equations (62) and (65) (velocity and frequency are easily confused, so caution is necessary).

$$v_{p,n} = \lambda_n \nu_n = \frac{h}{p_n} \frac{K_n}{h} = \frac{K_n}{p_n}. \tag{74}$$

Due to the above, the formula for the relativistic kinetic energy of the electron corresponding to Equation (61) is as follows.

$$K_{re,n} = -E_{re,n} = v_{p,n} p_{re,n}. \tag{75}$$

$$K_{re,n} = -E_{re,n} = m_n v_{g,n} v_{p,n}. \tag{76}$$

The energy of a photon is found as the product of the photon's momentum and the speed of light. The kinetic energy of an electron, in contrast, is determined by the product of the electron's momentum and its phase velocity.

Here, the phase velocity of the electron wave is derived with two methods by appropriately combining those formulas.

First,

$$v_{p,n} = \frac{K_{re,n}}{p_{re,n}} = \frac{m_n^2 v_{g,n}^2}{m_e + m_n} \frac{1}{m_n v_{g,n}} = \frac{m_n}{m_e + m_n} v_{g,n}. \tag{77}$$

The following relation is used here.

$$m_n = m_e \left(1 + \frac{\alpha^2}{n^2}\right)^{-1/2}. \tag{78}$$

When that is done, Equation (77) can be written as follows.

$$\begin{aligned} v_{p,n} &= \frac{m_n}{m_n \left[\left(1 + \frac{\alpha^2}{n^2}\right)^{1/2} + 1\right]} v_{g,n} \\ &= \left[\frac{n}{\left(n^2 + \alpha^2\right)^{1/2} + n}\right] v_{g,n}. \end{aligned} \tag{79}$$

Next, the following equation obtained from Equation (22) is used.

$$v_{g,n} = \frac{\alpha c}{n}. \tag{80}$$

Then,

$$v_{p,n} = \frac{nc}{\alpha}\left[\left(1 + \frac{\alpha^2}{n^2}\right)^{1/2} - 1\right]. \tag{81}$$

Equation (81) can also be written as follows.

$$v_{p,n} = \frac{c}{\alpha}\left[\left(n^2 + \alpha^2\right)^{1/2} - n\right]. \tag{82}$$

In the second method, the phase velocity is defined as follows.

$$v_{\mathrm{p},n}=\frac{c^2\left(m_{\mathrm{e}}-m_n\right)}{c\left(m_{\mathrm{e}}^2-m_n^2\right)^{1/2}}. \tag{83}$$

Rearranging this equation,

$$\begin{aligned} v_{\mathrm{p},n}&=\frac{c\left(m_{\mathrm{e}}-m_n\right)^{1/2}\left(m_{\mathrm{e}}-m_n\right)^{1/2}}{\left(m_{\mathrm{e}}-m_n\right)^{1/2}\left(m_{\mathrm{e}}+m_n\right)^{1/2}} \\ &=c\left(\frac{m_{\mathrm{e}}-m_n}{m_{\mathrm{e}}+m_n}\right)^{1/2}. \end{aligned} \tag{84}$$

Rearranging further,

$$\begin{aligned} v_{\mathrm{p},n}&=c\left[m_{\mathrm{e}}-\frac{m_{\mathrm{e}}}{\left(1+\alpha^2/n^2\right)^{1/2}}\right]^{1/2}\left[m_{\mathrm{e}}+\frac{m_{\mathrm{e}}}{\left(1+\alpha^2/n^2\right)^{1/2}}\right]^{-1/2} \\ &=c\left[\frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left(n^2+\alpha^2\right)^{1/2}+n}\right]^{1/2}. \end{aligned} \tag{85}$$

The following formula can be derived from Equations, (51) and (85).

$$v_{\mathrm{p},n}=c\left(\frac{r_{\mathrm{re},n}^{-}}{r_{\mathrm{re},n}^{+}}\right)^{1/2}. \tag{86}$$

Next, let us consider the kinetic energy of the electron.
First, from Equations (75) and (84),

$$K_{\mathrm{re},n}=v_{\mathrm{p},n}p_{\mathrm{re},n}=cp_{\mathrm{re},n}\left(\frac{m_{\mathrm{e}}-m_n}{m_{\mathrm{e}}+m_n}\right)^{1/2}. \tag{87}$$

Next, from Equations (75) and (85),

$$K_{\mathrm{re},n}=cp_{\mathrm{re},n}\left[\frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left(n^2+\alpha^2\right)^{1/2}+n}\right]^{1/2}. \tag{88}$$

Also, from Equations (75) and (86),

$$K_{\mathrm{re},n}=cp_{\mathrm{re},n}\left(\frac{r_{\mathrm{re},n}^{-}}{r_{\mathrm{re},n}^{+}}\right)^{1/2}. \tag{89}$$

Finally, we confirm the relationship between $v_{\mathrm{p},n}$ and $v_{\mathrm{g},n}$.
First, from Equation (77),

$$\frac{v_{\mathrm{p},n}}{v_{\mathrm{g},n}}=\frac{m_n}{m_{\mathrm{e}}+m_n}. \tag{90}$$

Next, from Equation (79),

$$\frac{v_{\mathrm{p},n}}{v_{\mathrm{g},n}}=\frac{n}{\left(n^2+\alpha^2\right)^{1/2}+n}. \tag{91}$$

Thus,

$$\frac{v_{\mathrm{p},n}}{v_{\mathrm{g},n}} \approx \frac{1}{2}. \tag{92}$$

Equation (92) does not perfectly match the value obtained from Equations (71) and (72). This topic will be left to the future.

7. Conclusions

The author first defined the following two formulas for the relativistic kinetic energy of an electron in a hydrogen atom.

$$K_{\mathrm{re},n} = m_{\mathrm{e}}c^2 - m_n c^2. \tag{93}$$

$$K_{\mathrm{re},n} = \frac{p_{\mathrm{re},n}^2}{m_{\mathrm{e}} + m_n}, \quad p_{\mathrm{re},n} = m_n v_n. \tag{94}$$

Equation (19), the energy-momentum relationship for an electron in a hydrogen atom, was derived from Equations (93) and (94). Then the following formula was derived by applying Equation (22) to Equation (19).

$$K_{\mathrm{re},n} = -E_{\mathrm{re},n} = m_{\mathrm{e}}c^2\left[1-\left(1+\frac{\alpha^2}{n^2}\right)^{-1/2}\right]. \tag{95}$$

The author used the relationship in Equation (22) when deriving Equation (96). However, the following relationship is naturally regarded as more important than Equation (22) [14].

$$\frac{p_{\mathrm{re},n}}{m_n c} = \frac{\alpha}{n}. \tag{96}$$

In Section 6, the following formula was derived for the relativistic kinetic energy of an electron in a hydrogen atom.

$$K_{\mathrm{re},n} = v_{\mathrm{p},n}\, p_{\mathrm{re},n}. \tag{97}$$

In Maxwell's theory, on the other hand, the energy of light is given by the following formula.

$$E = cp. \tag{98}$$

Equations (97) and (98) are extremely similar formulas. They can be regarded as formulas for the energies of particles and waves. The phase velocity of an electron wave is likely an important physical quantity on a par with the speed of light.

Also, Equation (97) can be described as follows.

$$K_{\mathrm{re},n} = m_n v_{\mathrm{g},n} v_{\mathrm{p},n}. \tag{99}$$

If the velocity of an applicable particle is low, then $m_{\mathrm{e}} \approx m_n$ so taking Equation (92) into account,

$$K_{\mathrm{re},n} = m_n v_{\mathrm{g},n} v_{\mathrm{p},n} \approx \frac{1}{2} m_n v_n^2 \approx \frac{1}{2} m_{\mathrm{e}} v_n^2 = K_{\mathrm{cl},n}. \tag{100}$$

The formula for the classical kinetic energy of an electron holds because Equation (99) holds.

The following summarizes the formulas, including the phase velocity of the electron.

First, from Equations (75) and (84),

$$K_{\mathrm{re},n} = v_{\mathrm{p},n} p_{\mathrm{re},n} = c p_{\mathrm{re},n} \left(\frac{m_{\mathrm{e}} - m_n}{m_{\mathrm{e}} + m_n} \right)^{1/2} . \quad (101)$$

Next, from Equations (75) and (85),

$$K_{\mathrm{re},n} = c p_{\mathrm{re},n} \left[\frac{\left(n^2 + \alpha^2\right)^{1/2} - n}{\left(n^2 + \alpha^2\right)^{1/2} + n} \right]^{1/2} . \quad (102)$$

Also, from Equations (75) and (86),

$$K_{\mathrm{re},n} = c p_{\mathrm{re},n} \left(\frac{r_{\mathrm{re},n}^{-}}{r_{\mathrm{re},n}^{+}} \right)^{1/2} . \quad (103)$$

Finally, the table below summarizes the formulas for the kinetic energy of an electron in a hydrogen atom derived by classical mechanics, the STR, and the author.

Classical mechanics	STR	This paper
$K_{\mathrm{cl}} = \frac{1}{2} m v^2$ $K_{\mathrm{cl}} = \frac{p^2}{2m}$	$K_{\mathrm{re}} = mc^2 - m_0 c^2$ $K_{\mathrm{re}} = \frac{p_{\mathrm{re}}^2}{m + m_0}$	$K_{\mathrm{re},n} = m_{\mathrm{e}} c^2 - m_n c^2$ $K_{\mathrm{re},n} = \frac{p_{\mathrm{re},n}^2}{m_{\mathrm{e}} + m_n}$ $K_{\mathrm{re},n} = m_{\mathrm{e}} c^2 \left[1 - \left(\frac{n^2}{n^2 + \alpha^2} \right)^{1/2} \right]$ $K_{\mathrm{re},n} = v_{\mathrm{p},n} p_{\mathrm{re},n}$ $K_{\mathrm{re},n} = m_n v_{\mathrm{g},n} v_{\mathrm{p},n}$ $K_{\mathrm{re},n} = c p_{\mathrm{re},n} \left(\frac{m_{\mathrm{e}} - m_n}{m_{\mathrm{e}} + m_n} \right)^{1/2}$ $K_{\mathrm{re},n} = c p_{\mathrm{re},n} \left[\frac{\left(n^2 + \alpha^2\right)^{1/2} - n}{\left(n^2 + \alpha^2\right)^{1/2} + n} \right]^{1/2}$ $K_{\mathrm{re},n} = c p_{\mathrm{re},n} \left(\frac{r_{\mathrm{re},n}^{-}}{r_{\mathrm{re},n}^{+}} \right)^{1/2}$

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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