

# Alternative Formulation of Quantum Mechanics

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## Abstract

For a particle at rest in macroscopic space that begins moving when energy is added, the equation  $E^2 = c^2 p^2 + E_0^2$  for Einstein's energy-momentum relationship represents the relationship between the particle's total energy and momentum, and rest mass energy. When the kinetic energy of the particle increases, so does its total energy.

However, things are different when electrons at rest approach the nuclei of hydrogen atoms—protons—thereby creating hydrogen atoms. An electron with rest mass energy  $E_0$  will reduce its total energy by emitting photons outside the atom but at the same time will gain kinetic energy. This paper derives the following relationship for an electron inside a hydrogen atom:  $E^2 + c^2 p^2 = E_0^2$ .

When establishing the coefficient for the Dirac equation, a relativistic wave equation, Dirac assumed that this equation satisfied the Klein-Gordon equation. However, the Klein-Gordon equation is a quantized equation of Einstein's relationship.

Thus, we attempt to discover a coefficient for the Dirac equation which satisfies this quantized relationship as is newly derived in this paper.

However, these are not intended to disaffirm the Dirac equation; rather, the equations in this paper with these discovered coefficients are presented as alternative forms of the Dirac equation.

## I. Introduction

Referring to classical quantum theory, the relationship between the total energy ( $E$ ) and kinetic energy of an electron ( $K$ ) inside a hydrogen atom is:

$$E_n = -\frac{1}{2} \left( \frac{1}{4\pi\epsilon_0} \right)^2 \frac{me^4}{\hbar^2} \cdot \frac{1}{n^2} \quad (1a)$$

$$= \frac{E_1}{n^2} \quad (1b)$$

$$= -\frac{K_1}{n^2}, \quad n = 1, 2, \dots, E_n < 0. \quad (1c)$$

Here,  $n$  is a principal quantum number. In this case, the total energy of the electron has a negative value (See Appendix A).

According to classical quantum theory, the total energy of a hydrogen atom is considered to be zero when the electron is separated from the atomic nucleus by a distance of infinity and remains at rest in that location. The total energy of Eq. (I.1a) is the value obtained from this perspective.

In classical mechanics, we emphasize the difference in energy, not the absolute energy.

However, according to quantum mechanics textbooks, the eigenvalue of the energy of a hydrogen atom as obtained from the Dirac equation, which is a relativistic wave equation, is as follows [1].

$$E = mc^2 \left[ 1 - \frac{\gamma^2}{2n^2} - \frac{\gamma^4}{2n^4} \left( \frac{n}{|k|} - \frac{3}{4} \right) \right]. \quad (2)$$

It is important to note that energy here is defined on an absolute scale. Because  $Z = 1$  in the case of a hydrogen atom,  $\gamma = e^2/\hbar c$ , ( $\gamma$  is the fine structure constant). If we ignore for the third term of this equation and define it as an approximation, Eq. (I.2) can be written as follows.

$$E = mc^2 - \frac{1}{2} \left( \frac{1}{4\pi\epsilon_0} \right)^2 \frac{me^4}{\hbar^2} \cdot \frac{1}{n^2} \quad (3a)$$

$$= mc^2 + E_n. \quad (3b)$$

Moreover,  $E$  of Eq. (B4) defines an absolute quantity, which includes the electron's rest mass energy (See Appendix B).

From this fact, in this paper, the total energy in absolute terms,  $E_{ab,n}$  for an electron inside a hydrogen atom is defined as below.

$$E_{ab,n} = E_0 + K_n + V(r_n) \quad (4a)$$

$$= E_0 + V(r_n)/2 \quad (4b)$$

$$= E_0 + E_n, \quad n = 1, 2, \dots, E_n < 0. \quad (4c)$$

To agree with  $n$  on the left side,  $n$  is added to  $K$  and  $r$  on the right side.

Here,  $E_{ab,n}$  is the total energy as defined in absolute terms when the principal quantum number is  $n$ . Also,  $E_n$  is the energy of a hydrogen atom in Bohr's solution, and we can clearly see that it is an approximate definition, as is shown by the Eq. (I.4c) ignoring the third term of Eq. (I.2).

When defined on an absolute scale, the total energy of a hydrogen atom is less than the electron's rest mass energy. This enables us to determine that  $E_{ab,n}^2 < E_0^2$ .

Incidentally, one of the most important relationships in the Special Theory of Relativity (STR) is as follows.

$$E^2 = c^2 p^2 + E_0^2. \quad (5)$$

Here,  $E$  is the total energy of an object or a particle.

If we assume the particle in this case is an electron, an electron at rest in an isolated system will begin moving when it absorbs external energy. Eq. (I.5) shows the relationship between the electron's total energy and its momentum and rest mass energy.

The following equation is presumed to be true when deriving Eq. (I.5) [3].

$$dE = vdp. \quad (6)$$

When a particle moves through macroscopic space, for an isolated system, as its velocity increases, the kinetic energy and hence total energy of the particle will increase. In classical mechanics, the increase of kinetic energy corresponds to the work done by external forces, and we have:

$$dK = Fdx \quad (7a)$$

$$= \frac{dp}{dt} dx \quad (7b)$$

$$= vdp. \quad (7c)$$

Also, in this situation, the particle's total energy and kinetic energy increase, but the increases are equal. That is,

$$dE = dK. \quad (8)$$

Eq. (I.6) can be subsequently derived from Eq. (I.7c) and Eq. (I.8).

Next, let us imagine an electron that is at rest an infinite distance in macroscopic space from

nucleus of a hydrogen atom—a proton—and is attracted by the proton's electrical force, creating a hydrogen atom. The electron emits photons outside the atom and reduces its total energy, but at the same time gains an amount of kinetic energy equal to the reduced amount of energy.

The following relationship can be derived from Eq. (A5c).

$$-dE = dK. \quad (9)$$

The following relationship can be subsequently derived from Eq. (I.7c) and Eq. (I.9).

$$-dE = vdp. \quad (10)$$

We start from Eq. (I.6) when deriving Einstein's relationship (I.5), but when deriving the relationship inside a hydrogen atom, we must start from Eq. (I.10).

## **II. Relationship between energy and momentum of an electron inside a hydrogen atom**

Referring to a STR textbook, we derive the energy-momentum relationship of an electron inside a hydrogen atom [4].

In classical mechanics,

$$m = \frac{p}{v}. \quad (1)$$

And, in STR,

$$m = \frac{E}{c^2}. \quad (2)$$

If, further, we suppose that Eq. (II.2) describes a universal equivalence of energy and inertial mass, we can combine Eqs. (II.1) and (II.2) into a single statement:

$$E = \frac{c^2 p}{v}. \quad (3)$$

Next, by multiplying the left and right sides of Eqs. (I.10) and (II.3), we obtain:

$$EdE = -c^2 p dp. \quad (4)$$

We integrate this:

$$E^2 = -c^2 p^2 + \text{const.} \quad (5)$$

We shall next determine the constant of integration and energy  $E$  for Eq. (II.5).

The constant of integration Eq. (II.5) should normally be determined through experimentation.

However, from the analogy of Eq. (I.5) of this discussion, the constant of integration Eq. (II.5) can be assumed to be  $E_0^2$ .

Next, in order to derive the energy-momentum relationship established inside a hydrogen, we must define the absolute quantity of the total energy of the electron.

In this paper, we use Eq. (I.4) for  $E$  in Eq. (II.5). This enables us to rewrite Eq. (II.5) as:

$$E_{\text{ab},n}^2 + c^2 p_n^2 = E_0^2, \quad (6)$$

$$\left( E_0 + \frac{V(r_n)}{2} \right)^2 + c^2 p_n^2 = E_0^2, \quad (7)$$

$$(E_0 + E_n)^2 + c^2 p_n^2 = E_0^2, \quad n=1,2,\dots, E_n < 0. \quad (8)$$

Eq. (I.4) is a nonrelativistic equation, although substituting this equation for one that is relativistic (II.5) raises doubts concerning the mixture of relativistic and nonrelativistic equations. However, Eq. (I.5) is normally considered a relativistic equation, and can even actually be derived without some kind of relativistic request being required.

This is the most general equation that can be applied to particles moving at nonrelativistic speeds. However, when describing those moving at nonrelativistic speeds, since the approximation  $E(v) \approx mc^2 + (1/2)mv^2$  is substituted, things add up even in the absence of Eq. (I.5).

Also, in the case of Eq. (II.6), the same logic is materialized. This equation represents the relationship between the energy and momentum of an electron in a system in which the energy levels are degenerating.

### III. Recalculating Expression (II.8)

We perform some tasks in this chapter to verify the accuracy of Expression (II.8) derived in the previous chapter.

First, we calculate the momentum  $p_n$  of an electron with an energy state having a principal quantum number  $n$  using classical quantum theory and the results of this paper. The following relationship exists between kinetic energy  $K_n$  and momentum  $p_n$  of an electron moving at a nonrelativistic speed and having an energy level with a principal quantum number  $n$ .

$$K_n \doteq \frac{p_n^2}{2m}. \quad (1)$$

By substituting the right side of Eq. (I.1a) for  $K_n$  of the above equation, we obtain the following.

$$\frac{1}{2} \left( \frac{1}{4\pi\epsilon_0} \right)^2 \frac{me^4}{\hbar^2} \cdot \frac{1}{n^2} \doteq \frac{p_n^2}{2m}. \quad (2)$$

By doing so, we obtain:

$$p_n \doteq \left( \frac{1}{4\pi\epsilon_0} \right) \frac{me^2}{n\hbar}. \quad (3)$$

We next derive  $p_n$  from (II.8). This can be rewritten as follows.

$$(E_0 + E_n)^2 + c^2 p_n^2 = \left[ E_0 - \frac{1}{2} \left( \frac{1}{4\pi\epsilon_0} \right)^2 \frac{me^4}{\hbar^2} \cdot \frac{1}{n^2} \right]^2 + c^2 p_n^2 \quad (4a)$$

$$= \left[ E_0 - \left( \frac{e^2}{4\pi\epsilon_0 \hbar c} \right)^2 \frac{E_0}{2n^2} \right]^2 + c^2 p_n^2 \quad (4b)$$

$$= \left( 1 - \frac{\alpha^2}{2n^2} \right)^2 E_0^2 + c^2 p_n^2 \quad (4c)$$

$$= E_0^2. \quad (4d)$$

By doing so, we obtain:

$$\left( 1 - \frac{\alpha^2}{2n^2} \right)^2 E_0^2 + c^2 p_n^2 = E_0^2. \quad (5)$$

However,  $\alpha$  here is the following fine structure constant.

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}. \quad (6)$$

By expanding Eq. (III.5), we obtain:

$$p_n^2 = \left( \frac{\alpha^2}{n^2} - \frac{\alpha^4}{4n^4} \right) (mc)^2. \quad (7)$$

Incidentally, because  $\alpha^4 = (5.325 \times 10^{-5})\alpha^2$ , if we now set  $\alpha^4/4n^4 \approx 0$ , Eq. (III.7) can be written as:

$$p_n \doteq \frac{\alpha mc}{n} \quad (8a)$$

$$= \left( \frac{1}{4\pi\epsilon_0} \right) \frac{me^2}{n\hbar}. \quad (8b)$$

As shown above, we find that  $p_n$  as derived from (II.8) is the same as the result derived from classical quantum theory (III.3). Thus, Expression (II.8) has been shown to be true for an electron inside a hydrogen atom.

#### **IV. Comparison of energy of a hydrogen atom as derived from the Dirac equation and from Expression (II.6)**

In this chapter, we first use the method from quantum mechanics textbooks to derive the energy levels of a hydrogen atom using Dirac's relativistic wave equation [5].

Then, we use Expression (II.6) of this paper to derive the energy of a hydrogen atom, and then compare the two.

Dirac approached the problem of finding a relativistic wave equation by starting from the hamiltonian form is as follows:

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = H\psi(\mathbf{r}, t). \quad (1)$$

The simplest hamiltonian that is linear in the momentum and mass term is as follows:

$$H = c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2. \quad (2)$$

By substituting this into Eq. (IV.1), we obtain:

$$(E - c\boldsymbol{\alpha} \cdot \mathbf{p} - \beta mc^2) \psi = 0. \quad (3)$$

We also substitute  $E$  and  $\mathbf{p}$  in Eq. (IV.3):

$$E \rightarrow i\hbar \frac{\partial}{\partial t}, \quad \mathbf{p} \rightarrow -i\hbar \nabla. \quad (4)$$

The result is the following quantized expression:

$$\left( i\hbar \frac{\partial}{\partial t} + i\hbar c\boldsymbol{\alpha} \cdot \nabla - \beta mc^2 \right) \psi = 0. \quad (5)$$

Terms that involve the electromagnetic potentials can be added to Eq. (IV.5) in a relativistic way by making the usual replacements as follows, where the particle described by the equation

has electric charge  $e$ .

$$c\mathbf{p} \rightarrow c\mathbf{p} - e\mathbf{A}, \quad E \rightarrow E - e\varphi. \quad (6)$$

Here,  $E$  and  $\mathbf{p}$  represent the operators of Eq. (IV.4).

In special cases like a central field (in which  $\mathbf{A} = 0$  and  $\varphi$  are spherically symmetric), because  $\mathbf{A}(\mathbf{r}, t) = 0$ ,  $\varphi(\mathbf{r}, t) = \varphi(r)$ , this enables us to obtain:

$$H = c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2 + V, \quad (7)$$

where  $V = e\varphi$ .

Represented in the same style as Eq. (IV.5), we obtain:

$$\left( i\hbar \frac{\partial}{\partial t} + i\hbar c\boldsymbol{\alpha} \cdot \nabla - \beta mc^2 - V \right) \psi = 0. \quad (8)$$

By performing the complex calculations based on this equation, we can obtain the following eigenvalue for the energy of a hydrogen atom. (See [5] for detailed calculation)

$$E = mc^2 \left[ 1 + \frac{\alpha^2}{(s + n')^2} \right]^{-1/2}, \quad n' = 0, 1, 2, \dots \quad (9)$$

Note: To follow the same notation as with other formulas,  $\gamma$  is has been replaced with  $\alpha$ .

Additionally, the correct quantum number for Eq. (IV.9) is:

$$s = \sqrt{k^2 - \alpha^2}, \quad n = n' + |k|. \quad (10)$$

The fine structure is made evident by expanding Eq. (IV.9) in powers of  $\alpha^2$ .

If we now substitute Eq. (IV.10) into Eq. (IV.9), and take the terms of order  $\alpha^4$ , we obtain:

$$E = mc^2 \left[ 1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4} \left( \frac{n}{|k|} - \frac{3}{4} \right) \right]. \quad (11)$$

$n$  is the total quantum number,  $n'$  is the radial quantum number, and  $|k|$  can take on positive integer values. Eq. (IV.11) shows that the energy increases with increasing  $|k|$ . This is the explanation as given in Schiff's textbooks.

When  $n' = 0$  in Eq. (IV.11), the contents of the third term in parentheses on the right side equal 1/4, so we can rewrite as:

$$E = mc^2 \left( 1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{8n^4} \right). \quad (12)$$

Next, we shall derive the energy of a hydrogen atom from Eq. (II.6). First, we can rewrite Eq. (II.6) as:



$$E_{ab,n} = (E_0^2 - c^2 p_n^2)^{1/2} \quad (13a)$$

$$= mc^2 \left( 1 - \frac{p_n^2}{m^2 c^2} \right)^{1/2}. \quad (13b)$$

Then, using the relationship from Eq. (III.8a), an approximation, we obtain:

$$E_{ab,n} = mc^2 \left( 1 - \frac{\alpha^2}{n^2} \right)^{1/2}. \quad (14)$$

Using the binomial theorem expansion here, we obtain:

$$E_{ab,n} = mc^2 \left( 1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{8n^4} - \dots \right). \quad (15)$$

The Dirac equation obtains the energy levels through complex calculations based on Eq. (IV.8). However, we can easily obtain the energy levels by starting from Eq. (IV.13a), which has been derived in this paper.

However, because Eq. (IV.13a) is not a quantized expression, the energy value of Eq. (IV.15) can only be obtained for the energy values of degenerative states.

But because we can derive the energy values from the third term of Eq. (IV.15), these are more accurate than the values predicted by Bohr's hypothesis.

Eq. (IV.13a) also a more precise definition of  $E_{ab,n}$ , while Eq. (I.4c) and Eq. (II.8) are approximations.

## V. Discussion

Because the solutions for the Dirac equation and for Eq. (IV.14) are the same, we can predict that both expressions are equivalent under a certain condition.

That condition of equivalency is when  $n'$  of Eq. (IV.9) is zero. When this is true,  $n/|k|=1$ , and the solutions for both expressions are the same, so the following equation must also be true:

$$\left( 1 + \frac{\alpha^2}{s^2} \right)^{-1} = 1 - \frac{\alpha^2}{n^2}. \quad (1)$$

By substituting  $s$  from left side of this equation for (IV.10), the left side of Eq. (V.1) becomes:

$$\left( 1 + \frac{\alpha^2}{s^2} \right)^{-1} = \frac{k^2 - \alpha^2}{k^2}$$

$$= 1 - \frac{\alpha^2}{n^2}. \quad (2)$$

Ultimately, the energy values for Eq. (IV.9) and Eq. (IV.14) are the same when  $n' = 0$  in Eq. (IV.9).

Incidentally,  $|k|$  can be written as:

$$|k| = j + \frac{1}{2}. \quad (3)$$

Thus, Eq. (IV.11) can be written as:

$$E = mc^2 \left[ 1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4} \left( \frac{n}{j + \frac{1}{2}} - \frac{3}{4} \right) \right]. \quad (4)$$

$n/(j+1/2) - 3/4$  is smallest here when  $n/(j+1/2) = 1$ , and the energy level to satisfy this condition is the same as the energy level of Eq. (IV.15). This energy level is the highest energy level of all available levels of all  $n$ . The corresponding energy levels are  $1S_{1/2}(n = 1, l = 0, j = 1/2)$ ,  $2P_{3/2}(n = 2, l = 1, j = 3/2)$  and  $3D_{5/2}(n = 3, l = 2, j = 5/2)$ .

The levels with their nonrelativistic classifications are shown in the following table: (See Table)

$n$	$n' (= n -  k )$	$k [= \pm(n - n')]$	$l (= 0, 1, \dots, n-1)$	$j (=  l \pm 1/2 )$	Energy levels
1	0	1	0	1/2	$1S_{1/2}$ ○
2	1	1	0	1/2	$2S_{1/2}$
2	1	-1	1	1/2	$2P_{1/2}$
2	0	2	1	3/2	$2P_{3/2}$ ○
3	2	1	0	1/2	$3S_{1/2}$
3	2	-1	1	1/2	$3S_{1/2}$
3	1	2	1	3/2	$3P_{3/2}$
3	1	-2	2	3/2	$3D_{3/2}$
3	0	3	2	5/2	$3D_{5/2}$ ○

**Table:** Energy levels of a hydrogen atom as obtained from Eq. (V.4). Energy levels marked with ○ are the same as the energy as obtained from Eq. (IV.15).

## VI. New Relational Expression Quantization

In this chapter, we shall attempt to quantize the newly obtained relationship (II.6).

The fact that when we perform quantization for Eq. (I.5), we obtain the following Klein-Gordon equation is evident.

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi = -\hbar^2 c^2 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) \psi + m^2 c^4 \psi. \quad (1)$$

This equation described the wave function in relativistic terms, but this interpretation was inconsistent with the interpretation according to the more commonly used Schrödinger equation.

Dirac surmised that a correct equation to resolve this shortcoming must take the following form [6].

$$i\hbar \frac{\partial}{\partial t} \psi = \left[ -i\hbar c \left( \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3} \right) + \beta mc^2 \right] \psi. \quad (2)$$

This is a rewritten form of Eq. (IV.5).

Then, because this equation must satisfy the Klein-Gordon equation, Dirac thought that all that was left was to determine the unknown coefficients  $\alpha_i$  and  $\beta$ .

Now, when we perform quantization on Eq. (II.6), we obtain the following.

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi = \hbar^2 c^2 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) \psi + m^2 c^4 \psi. \quad (3)$$

Because we derived Eq. (II.6) in this paper in addition to Eq. (I.5), we must determine that the Dirac equation coefficient also satisfies Eq. (VI.3).

To do so, in order to differentiate from the existing Dirac equation (VI.2), we assume that the expression derived in this paper has the coefficients  $\alpha'_i$  and  $\beta'$ .

$$i\hbar \frac{\partial}{\partial t} \psi = \left[ -i\hbar c \left( \alpha'_1 \frac{\partial}{\partial x_1} + \alpha'_2 \frac{\partial}{\partial x_2} + \alpha'_3 \frac{\partial}{\partial x_3} \right) + \beta' mc^2 \right] \psi. \quad (4)$$

Extracting only the operator from Eq. (VI.4), and making an equation by squaring both sides, we obtain

$$\begin{aligned} -\hbar^2 \frac{\partial^2}{\partial t^2} \psi = & \left[ -\hbar^2 c^2 \left( \alpha'_1 \frac{\partial}{\partial x_1} + \alpha'_2 \frac{\partial}{\partial x_2} + \alpha'_3 \frac{\partial}{\partial x_3} \right)^2 - i\hbar c \left( \alpha'_1 \frac{\partial}{\partial x_1} + \alpha'_2 \frac{\partial}{\partial x_2} + \alpha'_3 \frac{\partial}{\partial x_3} \right) \beta' mc^2 \right. \\ & \left. - \beta' mc^2 i\hbar c \left( \alpha'_1 \frac{\partial}{\partial x_1} + \alpha'_2 \frac{\partial}{\partial x_2} + \alpha'_3 \frac{\partial}{\partial x_3} \right) + \beta'^2 m^2 c^4 \right] \psi. \end{aligned} \quad (5)$$

Since the left side of this equation is the same as the Klein-Gordon equation and Eq. (VI.3),

the right side should finally be the same as the right side of Eq. (VI.3).

Next, expanding the right side of Eq. (VI.5), we obtain the following.

$$\begin{aligned}
-\hbar^2 \frac{\partial^2}{\partial t^2} \psi = & \left[ -\hbar^2 c^2 \left( \alpha_1'^2 \frac{\partial^2}{\partial x_1^2} + \alpha_2'^2 \frac{\partial^2}{\partial x_2^2} + \alpha_3'^2 \frac{\partial^2}{\partial x_3^2} \right) - \hbar^2 c^2 (\alpha_1' \alpha_2' + \alpha_2' \alpha_1') \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \right. \\
& - \hbar^2 c^2 (\alpha_2' \alpha_3' + \alpha_3' \alpha_2') \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_3} - \hbar^2 c^2 (\alpha_3' \alpha_1' + \alpha_1' \alpha_3') \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_1} \\
& - i\hbar mc^3 (\alpha_1' \beta' + \beta' \alpha_1') \frac{\partial}{\partial x_1} - i\hbar mc^3 (\alpha_2' \beta' + \beta' \alpha_2') \frac{\partial}{\partial x_2} \\
& \left. - i\hbar mc^3 (\alpha_3' \beta' + \beta' \alpha_3') \frac{\partial}{\partial x_3} + \beta'^2 m^2 c^4 \right] \psi. \tag{6}
\end{aligned}$$

In order to make Eqs. (VI.3) and (VI.6) the same, the coefficients  $\alpha_i'$  and  $\beta'$  must be a  $4 \times 4$  matrix satisfying the following conditions.

$$\left. \begin{aligned} \alpha_i'^2 &= -1 \\ \alpha_i' \alpha_j' + \alpha_j' \alpha_i' &= 0 \\ \alpha_i' \beta' + \beta' \alpha_i' &= 0 \\ \beta'^2 &= 1 \end{aligned} \right\}, \quad i, j = 1, 2, 3 (i \neq j). \tag{7}$$

The solution which satisfies these conditions and is a clean combination is as follows.

$$\begin{aligned}
\alpha_1' &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} & \alpha_2' &= \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \\
\alpha_3' &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} & \beta' &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{8}
\end{aligned}$$

Dirac, however, obtained the following for coefficients  $\alpha_i$  and  $\beta$ .

$$\begin{aligned}
\alpha_1 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} & \alpha_2 &= \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\mathbf{\alpha}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad \beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (9)$$

This coefficient was derived from the following conditions.

$$\left. \begin{array}{l} \alpha_i^2 = 1 \\ \alpha_i \alpha_j + \alpha_j \alpha_i = 0 \\ \alpha_i \beta + \beta \alpha_i = 0 \\ \beta^2 = 1 \end{array} \right\}, \quad i, j = 1, 2, 3 (i \neq j). \quad (10)$$

We have thus confirmed that  $\beta = \beta'$ .

Incidentally, because Eq. (VI.2) and Eq. (IV.5) are both Dirac equations that satisfy Eq. (VI.1), we can predict that the following will also satisfy Eq. (VI.3):

$$\left( i\hbar \frac{\partial}{\partial t} - i\hbar c \boldsymbol{\alpha} \cdot \nabla - \beta mc^2 \right) \psi = 0. \quad (11)$$

However, even when we imagine the existence of Eq. (VI.11), which is slightly different from Eq. (IV.5), and attempt to derive  $\alpha_i$  and  $\beta$  while presuming that this equation will satisfy Eq. (VI.3), we obtain the same coefficient values as (VI.8).

Because it is not necessary to use separate notation in this equation, we therefore use the following Dirac equation which only has a different coefficient.

$$\left( i\hbar \frac{\partial}{\partial t} + i\hbar c \boldsymbol{\alpha}' \cdot \nabla - \beta mc^2 \right) \psi = 0. \quad (12)$$

In this paper, rather than adding a hamiltonian term  $-V$  to the expression, we determine the equation by changing the coefficient.

When using classical mechanics to define the energy of a hydrogen atom, potential energy was the amount of energy needed to satisfy the law of conservation of energy. In Eq. (I.4b), potential energy contributes to the decrease and increase of  $E_0$ . Therefore, even if we do not include the term  $V$  into the equation, that energy can be represented in the equation by adjusting  $E_0$ .

Incidentally, for methods of handling equations in quantum mechanics, there are the Schrödinger representation and the Heisenberg representation. The former of these calculate

time change under a wave function (state), while the latter calculates using time-dependent operators and the states are constant. Eq. (VI.12) of this paper and the existing Eq. (IV.8) stand in contrast to each other like these two representations.

## VII. Conclusion

1. The result we obtained differs from Einstein's energy-momentum relationship.

In macroscopic space, we obtain:

$$E^2 = c^2 p^2 + E_0^2. \quad (\text{I.5})$$

However, in the space inside a hydrogen atom, we find that Eq. (II.6) holds true:

$$E_{ab,n}^2 + c^2 p_n^2 = E_0^2, \quad (\text{II.6})$$

A limit to the applicability of Einstein's energy-momentum relationship exists.

2. The Dirac equation which describes the energy of a hydrogen atom is given as the following equation:

$$\left( i\hbar \frac{\partial}{\partial t} + i\hbar c \boldsymbol{\alpha} \cdot \nabla - \beta mc^2 - V \right) \psi = 0. \quad (\text{IV.8})$$

Where:

$$\boldsymbol{\alpha}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{\alpha}_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\alpha}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad \beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (\text{VI.9})$$

Here, rather than changing the Dirac equation (IV.5) coefficient that is true in free space, the problem is solved by adding the term  $-V$  to the hamiltonian.

In this paper, however, we have shown that the energy of a hydrogen atom can be described by the following equation as well:

$$\left( i\hbar \frac{\partial}{\partial t} + i\hbar c \boldsymbol{\alpha}' \cdot \nabla - \beta m c^2 \right) \psi = 0. \quad (\text{VI.12})$$

Where:

$$\boldsymbol{\alpha}'_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{\alpha}'_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\alpha}'_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \quad (\text{VI.8})$$

Because the Dirac equation (IV.5) does not include potential energy, the term  $V$  has been newly added in Eq. (IV.8). However,  $V$  is already included within  $E_{ab,n}$  in Eq. (II.6). Therefore,  $V$  is not needed in Eq. (VI.12) to satisfy (II.6).

In this paper, the problem is solved not by adding  $-V$  to the Dirac equation (IV.5), but by changing the coefficient from  $\boldsymbol{\alpha}_i$  to  $\boldsymbol{\alpha}'_i$ .

However, these are not intended to disaffirm the Dirac equation; rather, the equations in this paper with these discovered coefficients are presented as alternative forms of the Dirac equation.

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Chapter IV was borrowed the QM textbook of Schiff. Logic from the textbook of French was also used in deriving Eq. (II.5). I wish to express my gratitude to Schiff and French.

## Appendix A

Let us review the energy of an electron inside a hydrogen atom. Let us suppose that an atomic nucleus is at rest because it is heavy, and consider the situation where an electron (electric charge  $-e$ , mass  $m$ ) is orbiting at speed  $v$  along an orbit (radius  $r$ ) with the atomic nucleus as its center.

An equation describing the motion is as follows:

$$\frac{mv^2}{r} = \frac{1}{4\pi\epsilon_0} \cdot \frac{e^2}{r^2}. \quad (1)$$

From this, we obtain:

$$\frac{1}{2}mv^2 = \frac{1}{2} \cdot \frac{1}{4\pi\epsilon_0} \frac{e^2}{r}. \quad (2)$$

Meanwhile, the potential energy of the electron is:

$$V(r) = -\frac{1}{4\pi\epsilon_0} \cdot \frac{e^2}{r}. \quad (3)$$

Since the right side of Eq. (A2) is  $-1/2$  times the potential energy, Eq. (A2) indicates:

$$V(r) = -2\left(\frac{1}{2}mv^2\right). \quad (4)$$

Therefore, the total electron energy:

$$E = \frac{1}{2}mv^2 + V(r) \quad (5a)$$

$$= -\frac{1}{2}mv^2 \quad (5b)$$

$$= -K. \quad (5c)$$

Also, the total energy of the electron is equal to half its potential energy.

$$E = \frac{1}{2}V(r). \quad (6)$$

The reason for the difference in potential energy and kinetic energy in Eq. (A4) is thought to be the photonic energy  $\hbar\omega$  released by the electron. Accordingly, we can establish the following law of energy conservation.

$$[V(r) + K] + \hbar\omega = 0. \quad (7)$$



## Appendix B

Gasiorowicz discusses the relativistic analog of Schrödinger for a bound (scalar) electron inside a hydrogen atom, which does include the rest mass energy of the electron in an attractive, central potential [2].

This equation is

$$\left( \frac{E}{\hbar c} + \frac{Ze^2}{4\pi\epsilon_0\hbar c r} \right)^2 \psi = -\nabla^2 \psi + \left( \frac{mc}{\hbar} \right)^2 \psi, \quad (1)$$

which is the operator version of Eq. (1.5) when a potential is included,

$$(E - V)^2 = c^2 p^2 + E_0^2. \quad (2)$$

The solution by solving for this Eq. (B1) did not agree with the actual energy level of the hydrogen atom. The reason proposed is that electrons are 1/2 spin particles and do not follow the Klein-Gordon equation. However, as a remaining problem, the left side of Eq. (B2) is as follows.

$$E - V = (K + V) - V \quad (3a)$$

$$= K. \quad (3b)$$

Thus,  $K^2 > E_0^2$ , or  $(p^2/2m)^2 > (mc^2)^2$ , but this kind of inequality should normally not be possible.

Here, let us surmise that  $E$  of Eq. (B2) is defined not as the  $E$  of Eq. (A5c) but instead as:

$$E = E_0 - K. \quad (4)$$

By substituting this  $E$  into Eq. (B2) and considering the relation to Eq. (A4), we obtain:

$$(E_0 + K)^2 = c^2 p^2 + E_0^2. \quad (5)$$

This equation is identical to Einstein's relationship. In the end, the total energy  $E$  of Eq. (B2) is the energy as defined by Eq. (B4).  $E$  of Eq. (B2) includes the electron's rest mass energy and is defined on an absolute scale.

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