Theoretical Prediction of the Size of a Proton and Revision of the Rydberg Formula

Koshun Suto

Abstract

For a particle at rest in macroscopic space that begins moving when energy is added, the equation $E^2 = c^2p^2 + E_0^2$ for Einstein's energy-momentum relation represents the relationship between the particle's total energy and momentum, and rest mass energy.

When the kinetic energy of the particle increases, so does its total energy.

However, things are different when electrons at rest approach the nuclei of hydrogen atoms— protons—thereby creating hydrogen atoms.

An electron with rest mass energy E_0 will reduce its total energy by emitting photons outside the atom but at the same time will gain kinetic energy.

This paper derives the following equation as an equation for an electron inside an atom.

$(E_0 + E_n)^2 + c^2p_n^2 = E_0^2$, $(n = 1,2, \cdots ,E_n < 0)$.

The physical quantity, thought to be the radius of a proton, can be naturally derived by substituting Bohr's quantum condition into this equation. This value is r_p = 0.705fm (where 1fm = 10^{-15}m).

This raises the need to revise the famous Rydberg formula to take into consideration the size of a proton. The following equation was derived by this paper as an equation to predict the wavelength of a photon emitted when an electron inside a hydrogen atom undergoes a transition.

$1/\lambda=R[4/(\alpha^2+4m^2) - 4/(\alpha^2+4n^2)]$, $(n = m+1,m+2,\cdots)$.

As was true at the beginning of the 20th century, the current-day theoretical value and actual value of the Balmer series spectral wavelength for a hydrogen atom are not entirely consistent. The photon wavelength as predicted by this paper is closer to the actual value than the value predicted by classical quantum theory.

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1. Introduction

We know today that the radius of all atomic nuclei R can be approximated by the formula $R = r_0A^{1/3}$. Here, A is atomic mass number and r_0 is about 1.2fm (1fm = 10^{-15}m).

Therefore, r_0 can be considered the radius of a proton, the nucleus of a hydrogen atom.

However, this proton radius is a predicted value based on equations obtained from scattering experiments of other atomic nuclei. We are not currently able to theoretically predict the size of a proton.

If we wish to theoretically predict the size of a proton, we predict that it will be necessary to use ideas and theories still unknown to us. However, it is important to avoid conflicts with quantum theory and the special theory of relativity when deriving new equations as well.

While keeping the above in mind, we shall attempt in this paper to theoretically predict the size of a proton.

First, one of the important relationships in the special theory of relativity is as follows.

$$E^2 + c^2 p^2 = E_0^2. \tag{1}$$

Here, E is the total energy of an object or a particle, and E_0 is the rest mass energy m_0c^2.

If we assume the particle in this case is an electron, an electron at rest in an isolated system will begin moving when it absorbs external energy. Eq. (1.1) shows the relationship between the electron's total energy and its momentum and rest mass energy.

The following equation is presumed to be true when deriving Eq. (1.1) [1].

$$dE = vdp. \tag{2}$$

When a particle moves through macroscopic space, for an isolated system, as its velocity increases, the kinetic energy and hence total energy of the particle will increase.

In classical mechanics, the increase of kinetic energy corresponds to the work done by external forces, and we have:

$$dK = Fdx \tag{3a}$$

$$= \frac{dp}{dt}dx \tag{3b}$$

$$= vdp. \tag{3c}$$

Also, in this situation, the particle's total energy and kinetic energy increase, but the increases are equal. That is,

$$dE = dK. \tag{4}$$

Eq. (1.2) can be subsequently derived from Eq. (1.3c) and Eq. (1.4).

We know that Eq. (1.1) can be derived by integrating Eq. (1.2).

Next, let us imagine an electron that is at rest an infinite distance in macroscopic space from nucleus of a hydrogen atom—a proton—and is attracted by the proton's electrical force, creating a hydrogen atom. The electron emits photons outside the atom and reduces its total energy, but at the same time gains an amount of kinetic energy equal to the reduced amount of energy.

Therefore, we can see that Eq. (1.4) is not true for an electron inside an atom. (See Appendix A)

2. Electron energy as described according to classical mechanics

Let us review the energy of an electron inside a hydrogen atom.

Let us suppose that an atomic nucleus is at rest because it is heavy, and consider the situation where an electron (electric charge *–e*, mass *m*) is orbiting at speed *v* along an orbit (radius *r*) with the atomic nucleus as its center.

An equation describing the motion is as follows:

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2}. \tag{1}$$

From this, we obtain:

$$\frac{mv^2}{2} = \frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 r}. \tag{2}$$

Meanwhile, the potential energy of the electron is:

$$V(r) = -\frac{e^2}{4\pi\varepsilon_0 r}. \tag{3}$$

Since the right side of Eq. (2.2) is –1/2 times the potential energy, Eq. (2.2) indicates:

$$V(r) = -2\left(\frac{mv^2}{2}\right). \tag{4}$$

Therefore, the total electron energy:

$$E = \frac{mv^2}{2} + V(r) \tag{5a}$$

$$= -\frac{mv^2}{2} \tag{5b}$$

$$= -K. \tag{5c}$$

Also, the total energy of the electron is equal to half its potential energy.

$$E = \frac{V(r)}{2}. \tag{6}$$

The reason for the difference in potential energy and kinetic energy in Eq. (2.4) is thought to be the photonic energy $\hbar\omega$ released by the electron. Accordingly, we can establish the following law of energy conservation.

$$\left[V(r) + K\right] + \hbar\omega = 0. \tag{7}$$

3. Electron Energy according to the Special Theory of Relativity

Let us consider a situation in which a single electron is at rest in a macroscopic space and thus holds only rest mass energy.

Let us assume that this electron at rest is attracted to the proton; in other words, it is attracted to the atomic nucleus of the hydrogen atom.

The electron tries to enter the region of the hydrogen atom. During this time, when the electron transit to a lower energy state and kinetic energy increases, an amount of energy equaling the increased kinetic energy is released outside the atom.

Considering these conditions and Eq. (2.7), we can establish the following relationship.

$$\left[E_0 + V(r) + K\right] + \hbar\omega = E_0. \tag{1}$$

This equation is Eq. (2.7) with the same amount of energy added to both sides and therefore, mathematically speaking, this relationship is always true for any amount of energy. Physically speaking, however, the addition of E_0 holds special significance.

In order to maintain the law of energy conservation, an energy source is needed to supply the increased kinetic energy and released photon energy.

A potential energy value is normally described in relative terms, but for an electron at rest in free space, the potential energy, in absolute terms, is zero.

The source of energy in Eq. (3.1) at first appears to be potential energy, but it seems unlikely that the physical quantity, which did not exist when the electron was originally at rest, has decreased.

Thus, in our discussion, in dealing with physical quantity, which in classical mechanics is called the potential energy of a hydrogen atom, we offer the hypothesis that this physical quantity corresponds to the reduction of the electron's rest mass energy.

When considered in this way, it is possible for the potential energy, which did not exist when the electron was at rest, to decrease.

When this decrease in energy is expressed as $-\Delta E_0$, we can establish the following two equations.

$$\begin{aligned} V(r) &= -\Delta E_0 \\ &= -(K + \hbar\omega). \end{aligned} \tag{2}$$

$$-\frac{\Delta V(r)}{2} = \Delta K. \tag{3}$$

Here, half of the reduction in potential energy is used in the form of work to increase the kinetic energy of the electron. Because the other half is emitted outside the atom as photonic energy, total energy decreases.

Meanwhile, based on Eq. (2.6), $\Delta V(r)/2$ is equal to the reduction in total energy. Namely,

$$\frac{\Delta V(r)}{2} = \Delta E. \tag{4}$$

(when $\Delta E < 0$)

Also, as evident from Eq. (2.5c), the following relationship can be derived from Eq. (3.3) and Eq. (3.4).

$$-dE = dK. \tag{5}$$

When work is performed against the electron inside a hydrogen atom and the kinetic energy of the electron increases, total energy decreases.

The following relationship can be subsequently derived from Eq. (1.3c) and Eq. (3.5).

$$-dE = vdp. \qquad (6)$$

4. Relationship between energy and momentum of an electron inside a hydrogen atom

Referring to a special theory of relativity textbook, we derive the energy-momentum relationship of an electron inside a hydrogen atom [2].

In classical mechanics,

$$m = \frac{p}{v}. \qquad (1)$$

And, in special relativity,

$$m = \frac{E}{c^2}. \qquad (2)$$

If, further, we suppose that Eq. (4.2) describes a universal equivalence of energy and inertial mass, we can combine Eqs. (4.1) and (4.2) into a single statement:

$$E = \frac{c^2 p}{v}. \qquad (3)$$

Next, by multiplying the left and right sides of Eqs. (3.6) and (4.3), we obtain:

$$EdE = -c^2 pdp. \qquad (4)$$

We integrate this:

$$E^2 = -c^2 p^2 + \text{const}. \qquad (5)$$

As shown by Eq. (1.1), an electron at rest has rest mass energy E_0. Similarly, when an electron at rest an infinite distance from a hydrogen atom is absorbed into an atom, the origination energy can be assumed to be E_0.

The constant of integration Eq. (4.5) should normally determined through experimentation.

However, from the analogy of Eq. (1.1) of this discussion, the constant of integration Eq. (4.5) can be assumed to be $E_0{}^2$. (See Appendix B)

Thus,

$$E^2 + c^2 p^2 = E_0^2. \qquad (6)$$

5. Total energy of an electron as defined from an absolute viewpoint

Referring to classical quantum theory and Eq. (2.5c), the relationship between the total energy and kinetic energy of an electron inside a hydrogen atom is:

$$E_n = -\frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{me^4}{\hbar^2}\cdot\frac{1}{n^2} \tag{1a}$$

$$= \frac{E_1}{n^2} \tag{1b}$$

$$= -\frac{K_1}{n^2}, \qquad (n = 1, 2, \cdots). \tag{1c}$$

Here, n is a principal quantum number. In this case, the total energy of the electron has a negative value.

Thus, the total energy of Eq. (5.1a) is not a value obtained from an absolute measurement.

In classical mechanics, we emphasize the difference in energy, not the absolute energy.

However, in order to derive the energy-momentum relationship established inside an atom, we must define the absolute quantity of total energy of the electron.

Fortunately, E of Eq. (B.4) defines an absolute quantity, which includes the electron's rest mass energy. Therefore, a definition of Eq. (B.4) is an important guideline for total energy as defined in this paper.

According to existing theory, the total energy of an electron is considered to be zero when the electron is separated from the atomic nucleus by a distance of infinity and remains at rest in that location. The total energy of Eq. (5.1a) is the value obtained from this perspective.

However, even if we place an electron at rest an infinite distance from its nucleus, the absolute energy of the electron is fundamentally not zero. According to Einstein, an electron in this state should have rest mass energy E_0 [3].

From this fact, Eq. (B.4), and Eq. (5.1a), in this paper, total energy in absolute terms, E_{ab}, for an electron inside a hydrogen atom is defined as below.

$$E_{\mathrm{ab},n} = E_0 + E_n, \tag{2}$$

(when, n=1,2,···, E_n< 0).

Here, $E_{\mathrm{ab},n}$ is the total energy as defined in absolute terms when the principal quantum number is n.

This definition can be used to rewrite Eq. (4.6) as:

$$(E_0 + E_n)^2 + c^2 p_n^2 = E_0^2, \tag{3}$$

(when, n=1,2,···, E_n< 0).

Eq. (5.2) is a non-relativistic equation, although substituting this equation for one that is relativistic (4.6) raises doubts concerning the mixture of relativistic and non-relativistic equations.

However, Eq. (1.1) is normally considered a relativistic equation, and can even actually be derived without some kind of relativistic request being required.

This is the most general equation that can be applied to particles moving at non-relativistic speeds. However, when describing those moving at non-relativistic speeds, since the approximation $E(v)\approx E_0+(1/2)(E_0/c^2)v^2$ is substituted, things add up even in the absence of Eq. (1.1).

Also, in the case of Eq. (4.6), the same logic is materialized. Thus, from Eqs. (4.6) and (5.2), we obtain Eq. (5.3).

Eq. (5.3) is Eq. (4.6), which includes the principal quantum number n. This equation represents the relationship between the energy and momentum of an electron in a system in which the energy level is degenerating.

6. Recalculating Expression (5.3)

We perform some tasks in this chapter to verify the accuracy of expression (5.3) derived in the previous chapter.

First, we calculate the momentum p_n of an electron with an energy state having a principal quantum number n using classical quantum theory and the results of this paper.

The following relationship exists between kinetic energy K_n and momentum p_n of an electron moving at a non-relativistic speed and having an energy level with a principal quantum number n.

$$K_n \doteq \frac{p_n^2}{2m}. \tag{1}$$

By substituting the right side of Eq. (5.1a) for K_n of the above equation, we obtain the following.

$$\frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{me^4}{\hbar^2}\frac{1}{n^2} \doteq \frac{p_n^2}{2m}. \tag{2}$$

By doing so, we obtain:

$$p_n \doteq \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{me^2}{n\hbar}. \tag{3}$$

We next derive p_n from the Eq. (5.3). This can be rewritten as follows.

$$(E_0+E_n)^2 + c^2 p_n^2 = \left[E_0 - \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{me^4}{\hbar^2}\cdot\frac{1}{n^2}\right]^2 + c^2 p_n^2 \tag{4a}$$

$$= \left[E_0 - \left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)^2 \frac{mc^2}{2n^2}\right]^2 + c^2 p_n^2 \tag{4b}$$

$$= \left(1 - \frac{\alpha^2}{2n^2}\right)^2 E_0^2 + c^2 p_n^2 \qquad \text{(4c)}$$

$$= E_0^2. \qquad \text{(4d)}$$

By doing so, we obtain:

$$\left(1 - \frac{\alpha^2}{2n^2}\right)^2 E_0^2 + c^2 p_n^2 = E_0^2. \qquad \text{(5)}$$

However, α here is the following fine structure constant.

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = 7.2973525376 \times 10^{-3}. \qquad \text{(6)}$$

By expanding Eq. (6.5), we obtain:

$$p_n^2 = \left(\frac{\alpha^2}{n^2} - \frac{\alpha^4}{4n^4}\right)(mc)^2. \qquad \text{(7)}$$

Incidentally, because $\alpha^4 = (5.325\times10^{-5})\alpha^2$, if we now set $\alpha^4/4n^2 \approx 0$, Eq. (6.7) can be written as:

$$p_n \doteq \frac{\alpha mc}{n} \qquad \text{(8a)}$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{me^2}{n\hbar}. \qquad \text{(8b)}$$

As shown above, we find that p_n as derived from Eq. (5.3) is the same as the result derived from classical quantum theory.

Thus, expression (5.3) has been shown to be true for an electron inside a hydrogen atom.

7. Orbit radius of an electron inside a hydrogen atom

In this chapter, we consider whether there is a new development for physics from Eq. (5.3).

According to classical quantum theory, the classical quantum radius r_n and energy E_n of a hydrogen atom are represented as follows:

$$r_n = \frac{4\pi\varepsilon_0 \hbar^2 n^2}{me^2} = a_B n^2, \qquad (n = 1, 2, \cdots). \qquad \text{(1)}$$

$$E_n = -\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 r_n}, \qquad (n = 1, 2, \cdots). \qquad \text{(2)}$$

Here, a_B is the Bohr radius. With this in mind and according to this theory, what value does r_n take?

The Bohr's quantum condition is represented as follows:

$$p_n \cdot 2\pi r_n = 2\pi n\hbar. \qquad \text{(3)}$$

If Eq. (6.8b) is substituted for p_n of Eq. (7.3) and Eq. (7.1) is substituted for r_n of equation (7.3), we obtain:

$$\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{me^2}{n\hbar}(2\pi)\frac{4\pi\varepsilon_0\hbar^2 n^2}{me^2}=2\pi n\hbar. \tag{4}$$

Because Eq. (6.8b) here is an approximate expression used with non-relativistic particles, if it is always equivalent to Eq. (7.4), then $a_{\rm B}$ may also be an approximate expression.

First, from Eq. (5.3), we find p_n as follows:

$$p_n=\frac{1}{c}(-2E_0E_n-E_n^2)^{\frac{1}{2}}. \tag{5}$$

Eq. (7.3) can also be expressed as:

$$p_n=\frac{n\hbar}{r_n}. \tag{6}$$

Substituting the value of Eq. (7.6) for momentum in Eq. (7.5), we obtain:

$$\frac{1}{c}(-2E_0E_n-E_n^2)^{\frac{1}{2}}\cdot 2\pi r_n=2\pi n\hbar. \tag{7}$$

Here, substituting the right side of Eq. (7.2) for E_n, we obtain:

$$\left[-2mc^2\left(-\frac{1}{2}\right)\frac{e^2}{4\pi\varepsilon_0 r_n}-\left(-\frac{1}{2}\right)^2\left(\frac{e^2}{4\pi\varepsilon_0 r_n}\right)^2\right]r_n^2=n^2\hbar^2c^2. \tag{8}$$

Solving for r_n in this equation, we obtain the following value:

$$r_n=\frac{1}{4}\frac{e^2}{4\pi\varepsilon_0 mc^2}+\frac{4\pi\varepsilon_0\hbar^2n^2}{me^2} \tag{9a}$$

$$=\frac{r_{\rm c}}{4}+a_{\rm B}n^2 \tag{9b}$$

$$=\frac{r_{\rm c}}{4}+\left(\frac{4\pi\varepsilon_0\hbar c}{e^2}\right)^2\frac{e^2}{4\pi\varepsilon_0 mc^2}\cdot n^2 \tag{9c}$$

$$=\left(\frac{1}{4}+\frac{n^2}{\alpha^2}\right)r_{\rm c},\qquad (n=1,2,\cdots). \tag{9d}$$

Here, $r_{\rm c}$ is the classic electron radius, which is defined by the following equation.

$$r_{\rm c}=\frac{e^2}{4\pi\varepsilon_0 mc^2}. \tag{10}$$

Ultimately, because Eq. (7.1) is an equation that can only be derived when E_n^2 of Eq. (7.5) is assumed to be zero, we can see that Eq. (7.1) is only an approximation.

8. Comparing the theoretical value and actual value of a photon's wavelength

In order to determine the validity of the Eq. (7.9b) derived in the previous chapter, in this chapter we derive an equation for the wavelength of a photon emitted when the electron of a hydrogen atom transitions from one energy level to another energy level. We then compare the photon wavelength predicted by this equation to existing theoretical and actual values.

Incidentally, E_n of Eq. (7.2) may be also written as follows.

$$E_n = -2\pi\hbar c R \frac{1}{n^2}, \qquad (n = 1, 2, \cdots). \tag{1}$$

Here, R is the Rydberg constant, which is defined by the following equation.

$$R = \frac{1}{4\pi}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{me^4}{c\hbar^3}. \tag{2}$$

The photon energy emitted during an transition between energy levels (E_n – E_m) and wavelength λ for principal quantum numbers m and n can be expressed as follows.

$$E_n - E_m = 2\pi\hbar c R\left(\frac{1}{m^2} - \frac{1}{n^2}\right), \qquad (n = m+1, m+2, \cdots). \tag{3}$$

$$\frac{1}{\lambda} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right). \tag{4}$$

Eq. (8.4) is the Rydberg formula. We next substitute the value from Eq. (7.9d) for r_n of Eq. (7.2).

$$E_n = -\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0}\frac{4\alpha^2}{r_c\left(\alpha^2 + 4n^2\right)} \tag{5a}$$

$$= -\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0}\frac{4\pi\varepsilon_0 mc^2}{e^2}\left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)^2 \frac{4}{\alpha^2 + 4n^2} \tag{5b}$$

$$= \frac{1}{4\pi}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{me^4}{c\hbar^3}\left(-2\pi\hbar c\right)\frac{4}{\alpha^2 + 4n^2} \tag{5c}$$

$$= -2\pi\hbar c R\left(\frac{1}{\left(\alpha^2/4\right) + n^2}\right). \tag{5d}$$

By doing so, the energy emitted when the principal quantum number transitions from n to m is expressed as follows.

$$E_n - E_m = 2\pi\hbar c R\left(\frac{1}{\left(\alpha^2/4\right) + m^2} - \frac{1}{\left(\alpha^2/4\right) + n^2}\right), \qquad (n = m+1, m+2, \cdots). \tag{6}$$

Wavelength λ of the photon corresponding to this energy is expressed as follows.

$$\frac{1}{\lambda}=R\left(\frac{1}{\left(\alpha^2/4\right)+m^2}-\frac{1}{\left(\alpha^2/4\right)+n^2}\right), \qquad (n=m+1,m+2,\cdots). \tag{7}$$

By adding this new term $r_c/4$ to Eq. (7.9b), the Rydberg formula (8.4) is revised into Eq. (8.7).

For the Balmer series of a hydrogen atom, m = 2. The actual and predicted values for the photon wavelength as obtained from Eq. (8.4) and Eq. (8.7) are summarized in the following table.

m	*n*	Quantum theoretical value [nm]	Actual value [nm]	Value predicted by this paper [nm]
2	3	656.112	656.273	656.115
2	4	486.009	486.133	486.011
2	5	433.937	434.047	433.938
2	6	410.070	410.174	410.072
2	7	396.907	397.007	396.909
2	8	388.807	388.905	388.809
2	9	383.442	383.539	383.444

Table 1. Comparison of the theoretical and actual values of the spectral wavelength of a Balmer series

For wavelength value calculations, the following values were used for R and α from CODATA (2006).

$$R = 10973731.568527\ \text{m}^{-1}. \tag{8}$$

$$\alpha = 7.2973525376\times 10^{-3}. \tag{9}$$

While the theoretical value and actual value for the spectral wavelength of a hydrogen atom are generally thought to be in complete agreement, the value predicted by this paper was closer to the actual value.

9. Conclusion

A. The result we obtained differs from Einstein's energy-momentum relationship.

In macroscopic space, we obtain Eq. (1.1):

$$E^2+c^2p^2=E_0^2.$$

However, in the space inside a hydrogen atom, we find that Eqs. (5.3) and (6.5) hold true:

$$(E_0 + E_n)^2 + c^2 p_n^2 = E_0^2,$$
(when, n=1,2,⋯, E_n< 0).

$$\left(1 - \frac{\alpha^2}{2n^2}\right)^2 E_0^2 + c^2 p_n^2 = E_0^2.$$

A limit to the applicability of Einstein's energy-momentum relationship exists.

B. The term $r_c/4$, which is newly added to Eq. (7.9), is predominantly considered to be related to the atomic nucleus, or in other words, the proton radius. The radius of the proton r_p is as follows:

$$r_p = \frac{r_c}{4} = 0.705\,\text{fm}.$$
(where 1fm = 10^{-15}m)

We can see that the physical quantities that determine the size of a proton are electrical charge e and the electron's rest mass energy mc^2.

C. The radius of a hydrogen atom, energy levels, and the wavelength of a photon emitted during transition are normally expressed by Eq. (7.1), Eq. (7.2) and Eq. (8.4), respectively. However, the theoretical research of this paper enables these to be revised as follows.

$$r_n = \frac{r_c}{4} + a_B n^2 = \left(\frac{1}{4} + \frac{n^2}{\alpha^2}\right) r_c, \qquad (n = 1, 2, \cdots).$$

$$E_n = -2\pi\hbar c R \left(\frac{1}{(\alpha^2/4) + n^2}\right).$$

$$\frac{1}{\lambda} = R\left(\frac{1}{(\alpha^2/4) + m^2} - \frac{1}{(\alpha^2/4) + n^2}\right), \qquad (n = m+1, m+2, \cdots).$$

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Appendix A

Traditionally, Eq. (1.1), or Einstein's energy-momentum relationship, was thought to hold true even inside an atom and was included in quantum mechanics theory. This was shown by quantizing Eq. (1.1) to derive the Klein-Gordon equation from and subsequently deriving the Dirac equation from this equation.

Appendix B

Gasiorowicz discusses the relativistic analog of Schrödinger for a bound (scalar) electron inside a hydrogen atom, which does include the rest mass energy of the electron in an attractive, central potential [4].

This equation is

$$\left(\frac{E}{\hbar c}+\frac{Ze^2}{4\pi\varepsilon_0\hbar c}\frac{1}{r}\right)^2\psi=-\nabla^2\psi+\left(\frac{mc}{\hbar}\right)^2\psi, \qquad (1)$$

which is the operator version of Eq. (1.1) when a potential is included,

$$\left[E-V(r)\right]^2=c^2p^2+E_0^2. \qquad (2)$$

The solution by solving for this Eq. (B.1) did not agree with the actual energy level of the hydrogen atom. The reason proposed is that electrons are 1/2 spin particles and do not follow the Klein-Gordon equation.

However, as a remaining problem, the left side of Eq. (B.2) is as follows.

$$E-V(r)=\left[K+V(r)\right]-V(r) \qquad (3a)$$

$$=K. \qquad (3b)$$

Thus, $K^2 > E_0^2$, or $(p^2/2m)^2 > (mc^2)^2$, but this kind of inequality should normally not be possible.

Here, let us surmise that E of Eq. (B.2) is defined not as the E of Eq. (2.5a) but instead as:

$$E=E_0-K. \qquad (4)$$

By substituting this E into Eq. (B.2) and considering the relation to Eq. (2.4), we obtain:

$$\left(E_0+K\right)^2=c^2p^2+E_0^2. \qquad (5)$$

This equation is identical to Einstein's relation. In the end, total energy E of Eq. (B.2) is the energy as defined by Eq. (B.4). E of Eq. (B.2) includes the electron's rest mass energy and is defined on an absolute scale. This is strong evidence to validate Eq. (5.2) as has been newly defined in this paper.

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