Limit of application of special relativity and the size of a proton

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Abstract

According to quantum mechanics, relationships between the total energy E and kinetic energy K of an electron in a hydrogen atom are: $E=-K$

Therefore, in the space in an atom, when a moving electron increases in velocity, its kinetic energy increases but its total energy decreases.

From this fact, in the space in an atom, we need to revise the relational expression of special relativity: $E^2 = c^2p^2 + E_0^2$

In this paper, we will obtain a new relational expression which takes the place of that of special relativity. Further, when we calculate the radius of a hydrogen atom with the new relational expression, we can obtain not only Bohr radius but the size of an atomic nucleus (i.e. proton).

I. Introduction

One of the important equations of special relativity is:

$$E^2 = c^2p^2 + E_0^2 \tag{I.1}$$

E means total energy of an object or a particle. E_0 means rest mass energy.

However, we can apply Eq.(I.1) not only to the relativistic motion but to the non-relativistic motion, because this relational expression shows the relationships between momentum and energies of an object.

When a particle moving in the macroscopic space, as an isolated system, increases in velocity, its kinetic energy and hence total energy will increase.

From this, we can obtain $E>E_0$.

In regard to an electron in an atom, however, when it increases in velocity its kinetic energy increases but its total energy decreases. From this it is predictable that in an atom, $E<E_0$.

In the next chapter we consider whether Eq.(I.1), which we can obtain in the macroscopic space, can be obtained in an atom.

II. Relationships between energy and momentum of an electron in a hydrogen atom

First of all, let us review the energy of an electron in a hydrogen atom.

Suppose the atomic nucleus is at rest because it is heavy. Consider the situation where electron (electric charge $-e$, mass m) is circulating at speed v along the orbit (radius r) whose center is the atomic nucleus.

We can obtain an equation as follows:

$$mv^2/r = e^2/(4\pi\varepsilon_0 r^2) \qquad \text{(II.1)}$$

Therefore:

$$mv^2/2 = (1/2)e^2/(4\pi\varepsilon_0 r) \qquad \text{(II.2)}$$

Meanwhile, the potential energy $V(r)$ of electron is:

$$V(r) = -e^2/(4\pi\varepsilon_0 r) \qquad \text{(II.3)}$$

Since the right side of Eq.(II.2) is $-1/2$ times of the potential energy, we have:

$$2(mv^2/2) = V(r) \qquad \text{(II.4)}$$

Therefore, relationships between the total energy E, kinetic energy K and potential energy are:

$$\begin{aligned} E &= V(r) + K \\ &= -2K + K \\ &= -K \\ &= V(r)/2 \end{aligned} \qquad \text{(II.5)}$$

It is known that, as to an electron, the absolute value of the total energy and that of the kinetic energy are equal to each other.

The total energy of an electron is equal to the half of the potential energy.

On the basis of the equation above, we will obtain relationships between energy and momentum of an electron in a hydrogen atom referring to a textbook on special relativity.[1]

In classical mechanics, the increment of kinetic energy corresponds to the work done by external forces, and we have

$$dK = Fdx = (dp/dt)dx = vdp \qquad \text{(II.6)}$$

In regard to a particle moving in the macroscopic space, the increment of total energy and that of kinetic energy is equal if potential energy doesn't change, i.e.,

$$dE = dK \qquad \text{(II.7)}$$

Where

$$dE = vdp \qquad \text{(II.8)}$$

In regard to an electron moving in an atom, however, the decrement of total energy is equal to the increment of kinetic energy, as is clear from Eq.(II.5):

$$-dE = dK \qquad \text{(II.9)}$$

From this and Eq. (II.6), we have

$$dE = -vdp \qquad \text{(II.10)}$$

We can obtain Eq.(I.1) from Eq.(II.8). However, the relationships between the energy and momentum of an electron in a hydrogen atom have to be obtained from Eq.(II.10).

In classical mechanics,

$$m = p/v \tag{II.11}$$

Also, from special relativity,

$$m = E/c^2 \tag{II.12}$$

From Eq.(II.11) and Eq.(II.12):

$$E = c^2 p/v \tag{II.13}$$

Next, we multiply together the left and right sides of the two equations (II.10) and (II.13):

$$EdE = -c^2 pdp \tag{II.14}$$

Then, we integrate this:

$$E^2 = -c^2 p^2 + E_0^2 \tag{II.15}$$

where E_0^2 is a constant of integration, written explicitly as the square of some constant energy.

Ⅲ. The meaning of energy *E*

What does energy E mean in Eq.(II.15)?

In classical mechanics, we are never concerned with absolute energy but only with energy difference.

In this paper, however, we consider absolute quantity of an electron's energy.

In existing theory, it is when we separate an electron from an atomic nucleus infinitely and put it at rest that we regard total energy of an electron as zero. Total energy in Eq.(II.5) is the value obtained from that point of view.

However, even if we put an electron at rest in an infinitely distant place, the absolute energy of an electron is essentially not zero. An electron should have rest mass energy E_0.

Considering this and Eq.(II.5), we define total energy of an electron in an atom E_{ab} in absolute sense as below:

$$\begin{aligned} E_{ab} &= E_0 + V(r) + K \\ &= E_0 - 2K + K \\ &= E_0 - K \\ &= E_0 + V(r)/2 \end{aligned} \tag{Ⅲ.1}$$

We regard E in Eq.(II.15) as total energy E_{ab} in Eq.(Ⅲ.1), not as total energy in (II.5).

Then, Eq.(II.15) is expressed as:

$$E_{ab}^2 + c^2p^2 = E_0^2 \tag{Ⅲ.2}$$

This equation shows the relationships between energy and momentum of an electron in a hydrogen atom. We have got a different result from special relativity's formula (I.1).

IV. Orbital radius and energy of an electron in a hydrogen atom

In this chapter we will consider whether we will have a new advance in physics from Eq.(Ⅲ.2) that we obtained in the previous chapter.

According to classical quantum theory, classical electron radius a_n and energy E_n of a hydrogen atom is given in the following:

$$a_n = \{4\pi\varepsilon_0 (h/2\pi)^2/(m_0 e^2)\}\, n^2 \qquad \text{(IV.1)}$$

$$E_n = -e^2/(8\pi\varepsilon_0 a_n) \qquad (n = 1,\ 2,\ \ldots) \qquad \text{(IV.2)}$$

Here, n is a principal quantum number.

From the view point of this paper, what will we get as the value of a_n and E_n ?

First, from total energy of an electron defined in Eq.(III.1) and from Eq.(III.2);

$$(E_0 - K)^2 + c^2p^2 = E_0^2 \qquad \text{(IV.3)}$$

Therefore

$$p = (2E_0K - K^2)^{1/2}/c \qquad \text{(IV.4)}$$

By the way, Bohr's quantum condition is;

$$p\, 2\pi r_n = 2\pi n\, (h/2\pi) \qquad \text{(IV.5)}$$

Substituting the value in Eq.(IV.4) for the momentum of Eq.(IV.5);

$$(2E_0K - K^2)^{1/2} r_n/c = n(h/2\pi) \qquad \text{(IV.6)}$$

Squaring both sides and substituting the value of the right side of Eq.(II.2) for the kinetic energy;

$$[2m_0c^2(1/2)e^2/(4\pi\varepsilon_0 r_n) - (1/2)^2\{e^2/(4\pi\varepsilon_0 r_n)\}^2]r_n^2 = n^2c^2(h/2\pi)^2 \qquad \text{(IV.7)}$$

Solving this referring to r_n ;

$$\begin{aligned} r_n &= (1/4)e^2/(4\pi\varepsilon_0 m_0c^2) + \{4\pi\varepsilon_0(h/2\pi)^2/(m_0e^2)\}n^2 \\ &= (r_e/4) + (1/r_e)(\lambda_c/2\pi)^2n^2 \\ &= (r_e/4) + n^2a_B \end{aligned} \qquad \text{(IV.8)}$$

Here, r_e is classical electron radius, and λ_c is compton wave length of electrons. They are given in the following;

$$r_e = e^2/(4\pi\varepsilon_0 m_0c^2) \qquad \text{(IV.9)}$$

$$\lambda_c = h/(m_0c) \qquad \text{(IV.10)}$$

To the radius r_n obtained in this paper[(IV.8)], the term $r_e/4$ is added besides the value obtained from classical quantum theory (IV.1).

Further, when n =1. the radius is:

$$r_1 = (r_e/4) + a_B \qquad \text{(IV.11)}$$

(Note that a_B is Bohr radius)

In addition, substituting r_n in Eq.(IV.8) for Eq.(IV.2):

$$E_n = -\, e^2/(8\pi\varepsilon_0 r_n) \qquad (n = 1,\ 2,\ \ldots) \qquad \text{(IV.12)}$$

V. Conclusion

1. In the macroscopic space we can obtain Eq.(I.1) of special relativity. In the space of a hydrogen atom, however, we can obtain Eq.(III.2).

2. It is natural to think that the term $r_e/4$ added newly in Eq.(IV.8) has something to do with the size of an atomic nucleus (i.e. proton).

In this paper, the radius of a proton r_p is: $r_p=7.05\times10^{-16}$ m.

At present, the radius of the nucleus of a hydrogen atom predicted from the experimental value is 1.2×10^{-15}m. It is possible to think that the reason for the difference between the two values is that the nucleus of a hydrogen atom contains no neutron.

In addition, from Eq.(IV.11), we have found atomic orbital radius that Bohr's classical quantum theory predicts is not the distance from the center of an atomic nucleus to the orbit, but the distance from the surface of an atomic nucleus to the orbit.

In Bohr radius the size of the atomic nucleus is not taken into consideration.

3. When an electron with the rest mass energy E_0 outside an atom is taken in an atom, the total energy of an electron decreases. Suppose the decreased energy is $-E'$. In this case, $E'/2$ is transformed into the kinetic energy of an electron, and the rest $E'/2$ is emitted as a photon outside an atom.

That is:

$$-E' = K + (h/2\pi)\,\omega = 0 \qquad \text{(V.1)}$$

(Note that $(h/2\pi)\,\omega$ is a photon's energy)

Further, considering Eq.(III.1), $-E'$corresponds to the potential energy of an electron.

That is:

$$-E' = V(r) \qquad \text{(V.2)}$$

Therefore

$$-E' = K + (h/2\pi)\,\omega + V(r) + K + (h/2\pi)\,\omega = 0 \qquad \text{(V.3)}$$

From this, we can predict the existence of the lower limit of an electron's total energy E_{ab} even in the classical mechanics.

From Eq.(III.1), the state where the potential energy has consumed all the rest mass energy is:

$$E_0 - 2K = 0 \qquad \text{(V.4)}$$

In this case, the kinetic energy of an electron becomes $m_0c^2/2$. Therefore, the relations between E (the total energy of an electron in existing theory) and E_{ab} is:

$$E = -\,m_0c^2/2 = -\,E_{ab} \qquad \text{(V.5)}$$

This is the minimum value of an electron's total energy considered in classical mechanics.

Substituting this for E_n in Eq.(IV.12), we can obtain the following value as the distance of closest approach.

$$r = e^2/(4\pi\varepsilon_0 m_0\, c^2) = r_e \qquad \text{(V.6)}$$

From Eq. (IV.8), the radius of an atomic nucleus can be considered $r_e/4$. Thus, in the prediction based on classical mechanics, it is clear that an electron is not absorbed in an atomic nucleus.

Furthermore, r_e agrees with the distance of closest approach of α -particle in Rutherford scattering.[2]

4. According to Copenhagen interpretation regarded as the conventional one on quantum mechanics, the microscopic particle, i.e. quantum, "behaves like a wave until its position is observed. But the moment its position is observed, its position as a particle is defined." However, we have obtained the size of a proton by calculation, not by experiment.

This means that a proton, a kind of quantum, is localized in a certain place as a particle, even if the position is not defined by observation.

By the way, in the famous two-slit interference experiment with electron, the conventional interpretation is as follows:

"An indivisible electron behaves as if it had come through both slits simultaneously."

However, this paper concludes as follows:

"Although an electron comes through either slit as a particle, the probability distribution of electrons found by the detector draws a pattern of interference in the end."

If the prediction of this paper is correct, Copenhagen interpretation ought to be revised.

5. Referring to the idea that L. de Broglie used when he predicted the existence of the matter wave, it becomes possible to discuss the magnitude of an electron, which is considered a particle without magnitude.

In this paper, we have found that the mass of an electron m_e is concerned in the magnitude of a proton. Supposing the mass of a proton m_p is concerned in the magnitude of an electron, the radius of an electron r_{el} is as follows:

$$\begin{aligned} r_{el} &= (1/4)e^2/(4\pi\varepsilon_0 m_p c^2) \\ &= r_p m_e/m_p = 3.84 \times 10^{-19}\ \text{m}. \end{aligned} \tag{V.7}$$

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References

[1] A.P.FRENCH, "Special Relativity" (THE M.I.T INTRODUCTORY PHYSICS SERIES) W.W.NORTON&COMPANY,New York.London,1968.

[2] E.RUTHERFORD,phil.Mag. **37** 537,1919