Limitation of Applicability of Einstein's Energy-Momentum Relationship

Koshun Suto

Abstract

When a particle moves through macroscopic space, for an isolated system, as its velocity increases, the kinetic energy (K) and hence total energy (E) of the particle will increase.

However, according to classical quantum theory, when the momentum and kinetic energy of an electron inside a hydrogen atom increases, total energy decreases.

From this truth, it is evident that the equation $E^2=c^2p^2+E_0^2$ for Einstein's energy-momentum relationship does not hold true inside a hydrogen atom.

In this paper, we will draw a following relationship: $(E_0+E_n)^2+c^2p_n^2 = E_0^2$ ($n=1,2,\cdots, E_n<0$)

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1. Introduction

Generally, physics theory that best describes the behavior of electrons within atoms is thought to be quantum mechanics.

This paper does not disagree, but further sets up the question of whether Einstein's energy-momentum relationship holds true even in the space inside atoms, and further attempts to answer to this question. (See Appendix A)

This question should have been first posed and answered when Einstein announced the abovementioned relationship – that is, when quantum mechanics was still incomplete (around the 1920s).

From this background, we shall consider as valid the discussion of this paper which does not rely upon complete quantum mechanics.

One of the important relationships in the Theory of Special Relativity is as follows.

$$E^2 = c^2p^2 + E_0^2 \tag{1.1}$$

E is the total energy of an object or particle, and E_0 is the rest mass energy m_0c^2.

Let us consider a situation in which a single electron is at rest within a macroscopic space and thus holds only rest mass energy. When external energy is added to this electron, it begins moving within the macroscopic space and the kinetic energy (K) and total energy (E) increase.

This relationship is described in the Eq.(1.1).

From this, we are able to obtain:

$$E^2 > E_0^2 \tag{1.2}$$

However, the situation differs when the electron at rest is located within an atom. When the former emits light outside the atom, it transits to a lower energy state. Here, although the electron gains kinetic energy, its total energy decreases and this condition cannot be described by Einstein's relationship (1.1).

According to classical quantum theory, however, the relationship between the total energy and kinetic energy of an electron inside a hydrogen atom is:

$$E_n = \frac{E_1}{n^2} = -\frac{K_1}{n^2} \qquad (n=1,2,\cdots) \tag{1.3}$$

Here, n is a principal quantum number.

In this case, the total energy of the electron has a negative value. (See Appendix B)

Thus, the total energy of Eq.(1.3) is not a value obtained from an absolute measurement.

In classical mechanics, we emphasize the difference in energy, not the absolute energy.

In this paper, however, we consider the absolute quantity of the energy of an electron.

According to existing theory, the total energy of an electron is considered to be zero when the electron is separated from the atomic nucleus by a distance of infinity and remains at rest in that location. The total energy of Eq.(1.3) is the value obtained from this perspective.

However, even if we place an electron at rest an infinite distance from its nucleus, the absolute energy of the electron is fundamentally not zero. According to Einstein, an electron in this state should have rest mass energy E_0.

The electron tries to enter the region of the hydrogen atom. During this time, when the electron transit to a lower energy state and kinetic energy increases, an amount of energy equaling the increased kinetic energy is released outside the atom.

In order to maintain the law of energy conservation, an energy source is needed to supply the increased kinetic energy and released photon energy.

This paper predicts that the energy source is the electron's rest mass energy, E_0. The above points may be summarized according to the following table:

i	ii	iii	iv	v	vi	vii
a	Absorbed	$E = E_0 + K$	$K = \hbar\omega$	Established	None	—
b	Emitted	$E = E_0 - K$	$-K = -\hbar\omega$	Not established	$K + \hbar\omega$	Rest mass energy

Table. 1 Comparison of the electron energy inside and outside an atom

i. Electron at rest in an isolated system (rest mass energy E_0)
a.When the photon is absorbed at rest and motion begins in macroscopic space
b.When the photon is emitted from a resting state and absorbed into a hydrogen atom
ii. Photon energy $\hbar\omega$ transfer
iii. Electron total energy E
iv. Electron's gained kinetic energy K
v. Law of energy conservation
vi. Difference from the law of energy conservation
vii. Source of energy to compensate for the difference in vi

From this fact, in this paper, total energy in absolute terms, E_{ab}, for an electron inside a hydrogen atom is defined as below.

$$
\begin{aligned}
E_{\mathrm{ab}} &= E_0 + E_n \\
&= E_0 + \frac{E_1}{n^2}
\end{aligned}
$$

$$\text{(However, } n=1,2,\cdots,\ E_n < 0\text{)} \tag{1.4}$$

Measured on an absolute scale, the total energy of an electron inside a hydrogen atom is less than the electron's rest mass energy.

From this definition, drawing the following relationship is inevitable.

$$E_{\mathrm{ab}}{}^2 < E_0{}^2 \tag{1.5}$$

From this fact, for the space inside a hydrogen atom, we must revise Einstein's relationship (1.1).

Thus, in the next chapter, we will draw a new relationship for the inside of a hydrogen atom.

2. Relationship between energy and momentum of an electron inside a hydrogen atom

Referring to a Theory of Special Relativity textbook, we derive the energy-momentum relationship of an electron inside a hydrogen atom [2].

In classical mechanics, an increase in kinetic energy is corresponds to external work forces.

Thus,

$$dK = Fdx = \frac{dp}{dt}dx = vdp \tag{2.1}$$

Also, the increased amount of total energy and increased amount of kinetic energy of a particle moving through macro space is equivalent to no change in potential energy, or:

$$dE = dK \tag{2.2}$$

From this, we obtain:

$$dE = vdp \tag{2.3}$$

However, for an electron moving inside a hydrogen atom, the reduction in total energy according to Eq.(B.5) and the increase in kinetic energy are equal:

$$-dE_{\mathrm{ab}}=dK \tag{2.4}$$

From this equation and Eq.(2.1), we obtain:

$$dE_{\mathrm{ab}}= -vdp \tag{2.5}$$

We can draw Eq.(1.1) from Eq.(2.3), but the relationship for energy and momentum of an electron inside a hydrogen atom should be drawn from Eq.(2.5).

In classical mechanics,

$$m = \frac{p}{v} \tag{2.6}$$

And, in Special Relativity,

$$m = \frac{E}{c^2} \tag{2.7}$$

It is assumed that the relationship of Eq.(2.7) is true inside a hydrogen atom, namely:

$$m = \frac{E_{\mathrm{ab}}}{c^2} \tag{2.8}$$

From Eqs (2.6) and (2.8),

$$E_{\mathrm{ab}} = \frac{c^2 p}{v} \tag{2.9}$$

Next, by multiplying the left and right sides of Eqs (2.5) and (2.9), we obtain:

$$E_{ab}dE_{ab} = -c^2 pdp \tag{2.10}$$

We integrate this:

$$E_{ab}^2 + c^2p^2 = E_0^2 \tag{2.11}$$

Here, E_0^2 is an integral constant written explicitly as the squared sum of energy. This relation shows the relationship between energy and momentum of an electron inside a hydrogen atom.

Eq.(1.4) is a non-relativistic equation, although substituting this equation for one that is relativistic Eq.(2.11) leads to doubts concerning the mixture of relativistic and non-relativistic equations.

However, Eq.(1.1) is normally considered a relativistic equation, and can even actually be derived without some kind of relativistic request being required.

This is the most general equation that can be applied to particles moving at non-relativistic speeds. However, when describing those moving at non-relativistic speeds, since the approximation $E(v) \approx E_0 + (1/2)(E_0/c^2)v^2$ is substituted, things add up even in the absence of Eq.(1.1).

Also in the case of Eq.(2.11), the same logic is materialized.

Thus, from Eqs.(1.4) and (2.11), we obtain the following relationship:

$$(E_0+E_n)^2 + c^2p_n^2 = E_0^2 \qquad (n=1,2,\cdots, E_n < 0) \tag{2.12}$$

Now, E_0 and E_1 are expressed as follows:

$$E_0 = m_0c^2 \tag{2.13}$$

$$E_1 = -\frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{m_0e^4}{\hbar^2} \tag{2.14}$$

The following values can be obtained from these two equations:

$$E_1 = -\frac{\alpha^2 E_0}{2} \tag{2.15}$$

However, α is a fine structure constant and defined as below.

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \tag{2.16}$$

The following equation, on the other hand, describes the relationship between E_1 and E_n.

$$E_n = \frac{E_1}{n^2} \qquad (n=1,2,\cdots) \tag{2.17}$$

Thus, we can obtain the following value for E_n, based on Eqs.(2.15) and (2.17).

$$E_n = -\frac{\alpha^2 E_0}{2n^2} \tag{2.18}$$

Substituting this value in Eq.(2.12), we obtain the following equation:

$$\left(1 - \frac{\alpha^2}{2n^2}\right)^2 E_0^2 + c^2 p_n^2 = E_0^2 \qquad (n = 1, 2, \cdots) \tag{2.19}$$

This equation is an Eq.(2.11), which includes the principal quantum number n. This equation represents the relationship between electron energy and momentum in a system in which the energy level is degenerating.

3. Orbit radius of an electron inside a hydrogen atom

In this chapter, we consider whether there is a new development for physics from Eq.(2.12).

According to classical quantum theory, the classical quantum radius r_n and energy E_n of a hydrogen atom are represented as follows:

$$r_n = \frac{4\pi\varepsilon_0 \hbar^2 n^2}{m_0 e^2} \qquad (n = 1, 2, \cdots) \tag{3.1}$$

$$E_n = -\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 r_n} \qquad (n = 1, 2, \cdots) \tag{3.2}$$

With this in mind and according to this theory, what values do r_n and E_n take?

First, from Eq.(2.12), we find p_n as follows:

$$p = \frac{1}{c}(-2E_0E_n - E_n^2)^{1/2} \tag{3.3}$$

Moreover, the Bohr quantum condition is represented as follows:

$$p_n \times 2\pi r_n = 2\pi n\hbar \tag{3.4}$$

Substituting the value of Eq.(3.3) for momentum in Eq.(3.4), we obtain:

$$\frac{1}{c}\left(-2E_0E_n - E_n^2\right)^{1/2} r_n = n\hbar \tag{3.5}$$

Squaring both sides and using the relation of Eq.(1.3), we obtain:

$$\left\{-\frac{2E_0E_1}{n^2} - \left(\frac{E_1}{n^2}\right)^2\right\} r_n^{\ 2} = n^2\hbar^2c^2 \tag{3.6}$$

Here, substituting the right side of Eq.(B.2) for E_1, we obtain:

$$\left\{-2m_0c^2\left(-\frac{1}{2}\right)\frac{e^2}{4\pi\varepsilon_0 r_n} - \left(-\frac{1}{2}\right)^2\left(\frac{e^2}{4\pi\varepsilon_0 r_n}\right)^2\right\} r_n^{\ 2} = n^2\hbar^2c^2 \tag{3.7}$$

Solving for r_n in this equation, we obtain the following value:

$$\begin{aligned} r_n &= \frac{1}{4}\frac{e^2}{4\pi\varepsilon_0 m_0c^2} + \frac{4\pi\varepsilon_0\hbar^2n^2}{m_0e^2} \\ &= \frac{r_\mathrm{e}}{4} + \frac{1}{r_\mathrm{e}}\left(\frac{\lambda_\mathrm{c}}{2\pi}\right)^2 n^2 \\ &= \frac{r_\mathrm{e}}{4} + a_\mathrm{B}n^2 \end{aligned} \tag{3.8}$$

Here, r_e is the classic electron radius and λ_c is the electron Compton wavelength, as follows:

$$r_\mathrm{e} = \frac{e^2}{4\pi\varepsilon_0 m_0c^2} \tag{3.9}$$

$$\lambda_c = \frac{h}{m_0 c} \tag{3.10}$$

To the radius found in Eq.(3.8), we add $r_e/4$ in addition to the values derived from classical quantum mechanics.

Also, the radius when n=1 is as follows:

$$r_1 = \frac{r_e}{4} + a_B \tag{3.11}$$

However, a_B is the Bohr radius. (See Appendix C)

4. New Relational Expression Quantization

This paper has fulfilled its objective in the preceding chapter, but for future expression, we shall attempt to quantize the newly obtained relationship (2.11).

The fact that when we perform quantization for equation (1.1),

$$E \to i\hbar \frac{\partial}{\partial t} \quad , \quad p \to -i\hbar \nabla \tag{4.1}$$

we obtain the following Klein-Gordon equation is evident.

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi = -\hbar^2 c^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) \psi + m^2 c^4 \psi \tag{4.2}$$

This equation described the wave function in relativistic terms, but this interpretation was inconsistent with the interpretation according to the more commonly used Schrödinger equation.

Dirac surmised that a correct equation to resolve this shortcoming must take the following form [3].

$$i\hbar \frac{\partial}{\partial t} \psi = \left\{ -i\hbar c \left(\alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3} \right) + \beta\, mc^2 \right\} \psi \tag{4.3}$$

Then, because this new equation must satisfy the Klein-Gordon equation, Dirac thought that all that was left was to determine the unknown coefficients α_i and β.

Now, when we perform quantization on equation (2.11), we obtain the following.

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = \hbar^2 c^2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)\psi + m^2c^4\psi \tag{4.4}$$

Extracting only the operator from equation (4.3), and making an equation by squaring both sides, we obtain

$$\begin{aligned}-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = \Bigg\{ & -\hbar^2c^2\left(\alpha_1\frac{\partial}{\partial x_1}+\alpha_2\frac{\partial}{\partial x_2}+\alpha_3\frac{\partial}{\partial x_3}\right)^2 \\ & - i\hbar c\left(\alpha_1\frac{\partial}{\partial x_1}+\alpha_2\frac{\partial}{\partial x_2}+\alpha_3\frac{\partial}{\partial x_3}\right)\beta\, mc^2 \\ & -\beta\, mc^2 i\hbar c\left(\alpha_1\frac{\partial}{\partial x_1}+\alpha_2\frac{\partial}{\partial x_2}+\alpha_3\frac{\partial}{\partial x_3}\right)+\beta^2m^2c^4\Bigg\}\psi\end{aligned} \tag{4.5}$$

Since the left side of this equation is the same as the Klein-Gordon equation and equation (4.4), the right side should finally be the same as the right side of (4.4).

Next, expanding the right side of equation (4.5), we obtain the following.

$$\begin{aligned}-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = \Bigg\{ & -\hbar^2c^2\left(\alpha_1^2\frac{\partial^2}{\partial x_1^2}+\alpha_2^2\frac{\partial^2}{\partial x_2^2}+\alpha_3^2\frac{\partial^2}{\partial x_3^2}\right)-\hbar^2c^2\left(\alpha_1\alpha_2+\alpha_2\alpha_1\right)\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} \\ & -\hbar^2c^2\left(\alpha_2\alpha_3+\alpha_3\alpha_2\right)\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3}-\hbar^2c^2\left(\alpha_3\alpha_1+\alpha_1\alpha_3\right)\frac{\partial}{\partial x_3}\frac{\partial}{\partial x_1} \\ & -i\hbar mc^3\left(\alpha_1\beta+\beta\alpha_1\right)\frac{\partial}{\partial x_1}-i\hbar mc^3\left(\alpha_2\beta+\beta\alpha_2\right)\frac{\partial}{\partial x_2} \\ & -i\hbar mc^3\left(\alpha_3\beta+\beta\alpha_3\right)\frac{\partial}{\partial x_3}+\beta^2m^2c^4\Bigg\}\psi\end{aligned} \tag{4.6}$$

In order to make equations (4.4) and (4.6) the same, the coefficients α_i and β must be a 4×4 matrix satisfying the following conditions.

$$\left.\begin{aligned}&\alpha_i^2=-1\\ &\alpha_i\alpha_j+\alpha_j\alpha_i=0\\ &\alpha_i\beta+\beta\alpha_i=0\\ &\beta^2=1\end{aligned}\right\}\quad i,j=1,2,3(i\neq j) \tag{4.7}$$

The solution which satisfies these conditions and is a clean combination is as follows.

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad \alpha_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$
$$\alpha_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.8}$$

Dirac, however, obtained the following for coefficients α_i and β.

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \alpha_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$
$$\alpha_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.9}$$

ψ from the Klein-Gordon equation was a single component, but in order to develop an equation that includes this sort of matrix, the wave guide function ψ itself would need to have four components.

Namely,

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \tag{4.10}$$

The equation obtained in this paper, as in the Dirac equation, also is an equation in which the 4 conditions are entangled. However, the goal of this paper is not to discuss the significance of these conditions, and so that discussion will not be taken up here.

5. Conclusion

1. The result we obtained differs from Einstein's energy-momentum relationship.

In macroscopic space, we obtain Eq.(1.1):

$$E^2 = c^2p^2 + E_0^2$$

However, in the space inside a hydrogen atom, we find that Eqs.(2.12) and (2.19) hold true:

$$(E_0+E_n)^2+c^2p_n^2 = E_0^2 \qquad \text{(However, } n=1,2,\cdots, E_n<0\text{)}$$

$$\left(1-\frac{\alpha^2}{2n^2}\right)^2 E_0^2 + c^2p_n^2 = E_0^2 \qquad (n=1,2,\cdots)$$

A limit to the applicability of Einstein's energy-momentum relationship exists.

2. The term $r_e/4$, which is newly added to Eq.(3.8), is predominantly considered to be related to the atomic nucleus, or in other words, the proton radius. The size of the atomic nucleus, $r_e/2$, is thus 1.41×10^{-15} m. If the atomic nucleus size is set to a mass of A for the purposes of this discussion, this is normally expressed as $(1.2\times10^{-15})\mathrm{A}^{1/3}$ m, showing these values to be in agreement.

3. In this paper we obtained different values (4.8) for the coefficients of Dirac's equation.
However, these are not intended to disaffirm Dirac's equation, but instead equations in this paper with these discovered coefficients are thought to be another form of Dirac's equation.

4. Referring to the idea that L.de Broglie used when he predicted the existence of the matter wave, it becomes possible to discuss the size of an electron, which is considered a particle without extent.

In this paper, we have found that the mass of an electron m_e is concerned in the size of a proton. Supposing the mass of a proton m_p is concerned in the size of an electron, the radius of an electron r_{el} is as follows:

$$r_{el}=(1/4)e^2/(4\pi\varepsilon_0 m_p c^2)$$
$$=r_p m_e/m_p=3.84\times10^{-19}\ \mathrm{m} \qquad (5.1)$$

5. According to Copenhagen interpretation regarded as the conventional one on quantum mechanics, the microscopic particle, i.e. quantum, "behaves like a wave until its position is observed. But the moment its position is observed, its position as a particle is defined."

However, we have obtained the size of a proton by calculation, not by experiment.

This means that a proton, a kind of quantum, is localized in a certain place as a particle, even if the position is not defined by observation.

By the way, in the famous two-slit interference experiment with electron, the conventional interpretation is as follows:

"An indivisible electron behaves as if it had come through both slits simultaneously."

However, this paper concludes as follows:

"Although an electron comes through either slit as a particle, the probability distribution of electrons found by the detector draws a pattern of interference in the end."

If the prediction of this paper is correct, Copenhagen interpretation ought to be revised.

6. In this paper, in Appendix A, it was asserted that equation (A.2) does not hold true.

However, if we assume that the E in equation (A.2) is not the E of equation (1.3) but instead is the E_{ab} of equation (1.4), the situation changes.

In this case, the left side of equation (A.2) becomes:

$$\begin{aligned}(E-V)^2 &= (E_0-K-V)^2 \\ &= (E_0+K)^2 \end{aligned} \tag{5.2}$$

Thus, equation (A.2) can be expressed as follows.

$$(E_0+K)^2=c^2p^2+E_0^2 \tag{5.3}$$

This equation is Einstein's relationship (1.1) itself.

However, this equation is meaningful for the relationship between energy of an electron and momentum in free space, not for the relationship between energy of an electron and momentum within an atom.

Appendix A

Gasiorowicz discusses the relativistic analog of Schrödinger for a bound (scalar) electron in a hydrogen atom, which does include the rest mass energy of the electron in an attractive, central potential [1].

This equation is

$$\left(\frac{E}{\hbar c}+\frac{Ze^2}{4\pi\varepsilon_0\hbar c}\frac{1}{r}\right)^2\psi=-\nabla^2\psi+\left(\frac{mc}{\hbar}\right)^2\psi \tag{A.1}$$

which is the operator version of Eq.(1.1) when a potential is included,

$$(E-V)^2=c^2p^2+E_0^2 \tag{A.2}$$

The solution by solving for this Eq.(A.1) did not agree with the actual energy level of the hydrogen atom. The reason proposed is that electrons are 1/2 spin particles and do not follow the Klein-Gordon equation.

However, as a remaining problem, the left side of Eq.(A.2) is as follows.

$$\begin{aligned} E-V &= K+V-V \\ &= K \end{aligned} \tag{A.3}$$

Thus, $K^2 > E_0^2$, or $(p^2/2m)^2 > (m_0c^2)^2$, but this kind of inequality should normally not be possible. Therefore, in this paper we search for a relationship to take the place of Eq.(A.2).

Appendix B

Let us review the energy of an electron inside a hydrogen atom. (See Acknowledgement)

Let us suppose that the atomic nucleus is at rest because it is heavy, and consider the situation where an electron (electric charge $-e$, mass m) is orbiting at speed v along an orbit (radius r) with the atomic nucleus as its center.

A equation describing the motion is as follows:

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2} \tag{B.1}$$

From this, we obtain:

$$\frac{mv^2}{2} = \frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 r} \tag{B.2}$$

Meanwhile, the potential energy of the electron is:

$$V(r) = -\frac{e^2}{4\pi\varepsilon_0 r} \tag{B.3}$$

Since the right side of Eq.(B.2) is $-1/2$ times the potential energy, Eq.(B.2) indicates:

$$2\left(\frac{mv^2}{2}\right) = -V(r) \tag{B.4}$$

Therefore, the total electron energy:

$$\begin{aligned} E &= \frac{mv^2}{2} + V(r) \\ &= -\frac{mv^2}{2} \end{aligned} \tag{B.5}$$

is equal the value when expressed as kinetic energy.

Also, the total energy of the electron is equal to half its potential energy.

$$E = \frac{V(r)}{2} \tag{B.6}$$

Appendix C

On the other hand, we can obtain the following value for r_n, based on Einstein's relationship (1.1) and the Bohr quantum condition (3.4).

$$r_n = -\frac{r_{\mathrm{e}}}{4} + a_{\mathrm{B}} n^2 \tag{C.1}$$

It is difficult to say that this value is in agreement with the fact.

Acknowledgements

The Appendix B was borrowed and translated from the Japanese language textbook of Dr. H. Ezawa's. I wish to express my gratitude to Dr. H. Ezawa.

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