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Two Types of Space in the Hydrogen Atom not Predictable with Quantum Mechanics

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ABSTRACT

Einstein's energy-momentum relationship is not applicable to the electron in a hydrogen atom. Therefore, the author has previously derived an energy-momentum relationship applicable to the electron inside the hydrogen atom where potential energy exists. However, the initially-derived relationship did not incorporate the discontinuities in energy which are characteristic of quantum mechanics. Therefore, the author derived a new quantum condition to take the place of Bohr's quantum condition, i.e., $v_n / c = \alpha / n$**, and that was used to incorporate discontinuity into the relationship derived by the author. When that relationship is solved, it is evident that, in addition to the existing energy levels, there are also ultra-low energy levels where the electron mass becomes negative. A previously unknown state of the hydrogen atom exists, formed from an electron with negative mass and a proton with positive mass. The electron with negative mass exists near the proton. The author predicts that this unknown matter is the true nature of dark matter, an unknown source of gravity whose true nature is currently unknown.**

Keywords: Einstein's Energy-Momentum Relationship in a Hydrogen Atom, Relativistic Kinetic Energy, Potential Energy, Bohr's Quantum Condition, Dark Matter

INTRODUCTION

The most important conclusion derived from the special theory of relativity (STR) is the equivalence of inertial mass and energy [1]. Energy in all its forms has inertial mass [2]. To put it another way, all changes in the energy of an object ΔE correspond to changes in the object's inertial mass Δm [3]. Einstein expressed these as follows.

$$E = mc^2. \tag{1}$$

$$\Delta E = c^2 \Delta m. \tag{2}$$

In this paper, let us review the energy-momentum relationship of Einstein using a textbook [4].

Now, in classical mechanics, an increase in kinetic energy corresponds to work done by an external force. That is,

$$dE = Fdx = \frac{dp}{dt} dx. \tag{3}$$

Therefore,

$$dE = vdp. \tag{4}$$

The following relationship also holds in classical mechanics.

$$m = \frac{p}{v}. \tag{5}$$

In the textbook of French, the following equation is obtained by combining Equation (1) and Equation (5).

$$E = \frac{c^2 p}{v}. \tag{6}$$

Next, if the right-hand sides of Equation (4) and Equation (6), and the corresponding left hand sides, are multiplied together,

$$EdE = c^2 pdp. \tag{7}$$

Integrating this,

$$E^2 = c^2 p^2 + E_0^2. \tag{8}$$

Equation (8) can also be expressed as follows.

$$\left(mc^2\right)^2 = c^2 p^2 + \left(m_0 c^2\right)^2. \tag{9}$$

Here m_0 is rest mass and m is relativistic mass. Equation (8) is the energy-momentum relationship of Einstein that holds in an isolated system in free space.

If an object is at rest, $p = 0$ and thus Equation (8) is as follows.

$$E = m_0 c^2. \tag{10}$$

However, if we are satisfied with Equation (10) only, then the deeper meaning of the theory of relativity is lost. Typically, momentum is not zero. In that case, Equation (1) is used.

Equation (1) includes Equation (10) as a special case. In this paper, the energy in Equation (1) becomes important when dealing with the relativistic energy of the hydrogen atom.

Now, what sort of relation holds in the case of an electron in a hydrogen atom.

Let's consider a situation where an electron at rest in free space is taken into a hydrogen atom due to the electrostatic attraction of the atomic nucleus (proton).

When an electron at rest in free space is taken into an atom, the electron emits a photon. Also, the electron acquires a kinetic energy *K* equal to the energy $h\nu$ of the emitted photon. To ensure that the law of energy conservation holds in this situation, there must be a source which supplies these two types of energy.

In classical quantum theory, the energy source for these is thought to be the potential energy of the electron.

However, the true nature of the electron's potential energy is not discussed in classical quantum theory. That was a point the author was discontented with. Therefore, in order to substantively capture the potential energy of the electron, the author looked at the rest mass energy $m_e c^2$ of the electron. The author noticed that the reduction in electron rest mass energy of the electron $-\Delta m_e c^2$ corresponds to the potential energy of the electron, and that was expressed as follows [5][6].

$$V(r) = -\Delta m_e c^2. \tag{11}$$

Hence, the following law of energy conservation holds in an electron which has started moving.

$$K + h\nu - \Delta m_e c^2 = 0. \quad (12)$$

Half of the rest mass energy consumed by the electron changes into kinetic energy of the electron, and the other half is emitted outside the atom (when the electron is at rest at a position infinitely distant from the proton, the rest mass energy of the electron is not consumed, and thus the potential energy is zero. This corresponds to the previous definition of potential energy).

Since the total mechanical energy of the electron can be expressed by the sum of the kinetic energy K and potential energy $V(r)$,

$$E = K + V(r) = \frac{m_e v^2}{2} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}. \quad (13)$$

According to the Virial theorem, $2K = -V(r)$ in the case of a circular orbit, and thus the total mechanical energy can be written as follows.

$$E = -K = \frac{1}{2}V(r) = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}. \quad (14)$$

An electron in an atom acquires kinetic energy due to the emission of photon energy.

In contrast, if an electron is placed in an isolated system in free space, then the absorbed energy becomes the kinetic energy of the electron. For this reason, it is clear that Equation (9) does not hold inside the atom.

Based on the above, an energy-momentum relationship is derived that is applicable to an electron in a hydrogen atom.

ENERGY-MOMENTUM RELATIONSHIP APPLICABLE TO AN ELECTRON IN A HYDROGEN ATOM

The increase in kinetic energy of the electron corresponds to the work done with respect to the outside. This situation is the opposite of Equation (3), and thus Equation (3) must be rewritten as follows.

$$dE = -F dx = -\frac{dp}{dt}dx. \quad (15)$$

From this, we obtain not Equation (4) but rather

$$dE = -v dp. \quad (16)$$

Multiplying in the same way the right-hand sides of Equation (6) and Equation (16), and the corresponding left hand sides,

$$E dE = -c^2 p dp. \quad (17)$$

And integrating,

$$E^2 = -c^2 p^2 + E_0^2. \quad (18)$$

Equation (18) can also be expressed as follows.

$$\left(mc^2\right)^2 = -c^2 p^2 + \left(m_e c^2\right)^2, \quad m < m_e. \quad (19)$$

Equation (19) is an energy-momentum relationship applicable to an electron in a hydrogen atom which has potential energy [7].

Here, m is the relativistic mass of the electron. It must be noted that the mass of the electron in a hydrogen atom becomes smaller than m_e.

Incidentally, it is known that the following formula can be derived from Equation (9).

$$mc^2 = \frac{m_0 c^2}{\left(1 - v^2/c^2\right)^{1/2}}. \tag{20}$$

Using the same method, the following formula can be derived from Equation (19) [8].

$$mc^2 = \frac{m_e c^2}{\left(1 + v^2/c^2\right)^{1/2}}. \tag{21}$$

However, energy takes on continuous values in Equation (19). The discontinuity of energy characteristic of the micro world is not incorporated into this formula. Thus, discontinuous energy levels must be incorporated into Equation (19).

A QUANTUM CONDITION MORE SUBSTANTIAL THAN BOHR'S QUANTUM CONDITION

The energy levels derived by Bohr are given by the following formulas [9][10].

$$E_{\mathrm{BO},n} = -\frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{m_e e^4}{\hbar^2} \cdot \frac{1}{n^2} \tag{22a}$$

$$= -\frac{\alpha^2 m_e c^2}{2n^2}, \quad n = 1, 2, \cdots. \tag{22b}$$

Here, $E_{\mathrm{BO},n}$ signifies the energy levels derived by Bohr. Also, α is the fine-structure constant, and is defined as follows.

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = 7.2973525693 \times 10^{-3}. \tag{23}$$

If E in Equation (22b) is substituted into Equation (14), then the following formula can be derived as the orbital radius of the electron.

$$r_n = 4\pi\varepsilon_0 \frac{\hbar^2}{m_e e^2} n^2, \quad n = 1, 2, \cdots. \tag{24}$$

Bohr thought the following quantum condition was necessary to find the energy levels of the hydrogen atom.

$$m_e v_n \cdot 2\pi r_n = 2\pi n\hbar, \quad n = 1, 2, \cdots. \tag{25}$$

In Bohr's classical quantum theory, the energy of the hydrogen atom is treated non-relativistically, and thus here the momentum of the electron is taken to be $m_e v$. Also, the Planck constant h can be written as follows [11][12]:

$$\hbar = \frac{h}{2\pi} = \frac{m_e c \lambda_C}{2\pi}. \tag{26}$$

λ_C is the Compton wavelength of the electron.

When Equation (26) is used, the fine-structure constant α can be expressed as follows.

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{e^2}{2\varepsilon_0 m_e c^2 \lambda_C}. \tag{27}$$

Also, the classical electron radius r_e is defined as follows.

$$r_{\mathrm{e}} = \frac{e^2}{4\pi\varepsilon_0 m_{\mathrm{e}} c^2}. \tag{28}$$

If r_{e} / α is calculated here,

$$\frac{r_{\mathrm{e}}}{\alpha} = \frac{\lambda_{\mathrm{C}}}{2\pi}. \tag{29}$$

If Equation (24) is written using r_{e} and α, the result is as follows.

$$r_n = 4\pi\varepsilon_0 \frac{\hbar^2}{m_{\mathrm{e}} e^2} n^2 = \frac{e^2}{4\pi\varepsilon_0 m_{\mathrm{e}} c^2}\left(\frac{4\pi\varepsilon_0 \hbar c}{e^2}\right)^2 n^2 = \frac{r_{\mathrm{e}}}{\alpha^2} n^2. \tag{30}$$

Next, if $\hbar$ in Equation (26) and r_n in Equation (30) are substituted into Equation (25),

$$m_{\mathrm{e}} v_n \cdot 2\pi \frac{r_{\mathrm{e}}}{\alpha^2} n^2 = 2\pi n \frac{m_{\mathrm{e}} c \lambda_{\mathrm{C}}}{2\pi}. \tag{31}$$

If Equation (29) is also used, then Equation (31) can be written as follows.

$$m_{\mathrm{e}} v_n \cdot 2\pi \frac{r_{\mathrm{e}}}{\alpha^2} n^2 = 2\pi n \frac{m_{\mathrm{e}} c r_{\mathrm{e}}}{\alpha}. \tag{32}$$

From this, the following relationship can be derived [13].

$$\frac{v_n}{c} = \frac{\alpha}{n}. \tag{33}$$

Due to the above, it is evident that Equation (33) is contained in Equation (25). Here, n is the principal quantum number.

Substituting Equation (33) into Equation (21) here,

$$m_n = m_{\mathrm{e}}\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}. \tag{34}$$

Here, a subscript n was attached to m on the left side, to be like the subscript on the right side.

If both sides of this equation are multiplied by c^2 and both sides are squared,

$$\left(m_n c^2\right)^2 = \left(m_{\mathrm{e}} c^2\right)^2\left(\frac{n^2}{n^2+\alpha^2}\right). \tag{35}$$

Next, if Equations (33) and (34) are used,

$$\begin{aligned} p_n^2 &= m_n^2 v_n^2 \\ &= \frac{m_{\mathrm{e}}^2}{\left(1+\alpha^2/n^2\right)} \cdot \frac{\alpha^2 c^2}{n^2} \\ &= \left(m_{\mathrm{e}} c\right)^2\left(\frac{\alpha^2}{n^2+\alpha^2}\right). \end{aligned} \tag{36}$$

It is evident that Equation (19) can be written as follows using Equations (34) and (36).

$$\left(m_n c^2\right)^2 + c^2 p_n^2 = \left(m_{\mathrm{e}} c^2\right)^2. \tag{37}$$

$$\left(m_{\mathrm{e}} c^2\right)^2\left(\frac{n^2}{n^2+\alpha^2}\right) + c^2\left(m_{\mathrm{e}} c\right)^2\left(\frac{\alpha^2}{n^2+\alpha^2}\right) = \left(m_{\mathrm{e}} c^2\right)^2. \tag{38}$$

Equation (37) is an energy-momentum relationship applicable to an electron in a hydrogen atom which has potential energy.

Here, $m_n c^2$ is the relativistic energy of the electron.

Next, let's consider the meaning of Equation (33) contained in Bohr's quantum condition (25) by using Equations (37) and (38).

$$\frac{cp_n}{m_n c^2} = \frac{cm_n v_n}{m_n c^2} = \frac{v_n}{c}. \tag{39}$$

Here, referring also to Equation (38) when taking the ratio of the first and second term on the left side of Equation (37),

$$\frac{cp_n}{m_n c^2} = \frac{\alpha}{n}. \tag{40}$$

From this, the following relation can be derived.

$$\frac{cp_n}{m_n c^2} = \frac{v_n}{c} = \frac{\alpha}{n}. \tag{41}$$

Due to the above considerations, it is evident that Equation (33) is a more substantive quantum condition than Equation (25). Due to Equation (33), it is possible to identify discontinuous states that are permissible in terms of quantum mechanics in the continuous motions of classical theory. The author believes that it is more appropriate to bestow the status of quantum condition on Equation (33) than on Equation (25).

Past attempts to relativistically expand the energy levels of the hydrogen atom derived by Bohr have taken Equation (9) as their point of departure [14][15][16][17]. However, this is a mistake. The equation which treats an electron in a hydrogen atom relativistically is not the Dirac equation satisfying Equation (9). It must be another equation satisfying Equation (37). The author has already derived this equation [18][19].

Incidentally, it was once pointed out by Dirac that Equation (9) has a negative solution [20]. In the same way, the author has pointed out that Equation (37) has a negative solution [18]. The mass of an electron at negative energy levels becomes negative.

In the current universe, there is thought to exist a tremendous mass whose true nature is unknown (an unknown source of gravity). The author has presented matter formed from an electron with negative mass and a proton (atomic nucleus) with positive mass as a strong candidate for this unknown matter, i.e., dark matter [21-26].

Now, if the negative solution of Equation (37) is also incorporated, then the relativistic energy $E_{\mathrm{re},n}^{\pm}$ of a hydrogen atom can be written as follows [25].

$$E_{\mathrm{re},n}^{\pm} = \pm m_n c^2 = \pm m_{\mathrm{e}} c^2 \mp K_{\mathrm{re},n} = \pm m_{\mathrm{e}} c^2 \pm E_{\mathrm{SU},n} = \pm m_{\mathrm{e}} c^2 \left(\frac{n^2}{n^2 + \alpha^2} \right)^{1/2}, \quad n = 0, 1, 2, \cdots. \tag{42}$$

However, relativistic kinetic energy $K_{\mathrm{re},n}$ is defined here as follows [13].

$$K_{\mathrm{re},n} = -E_{\mathrm{SU},n} = \frac{p_n^2}{m_{\mathrm{e}} + m_n}. \tag{43}$$

Also, $E_{\mathrm{SU},n}$ signifies the total mechanical energy predicted by this paper (Suto).

If, the relation between the energy levels $E_{\mathrm{SU},n}$ and the relativistic energy levels $E_{\mathrm{re},n}$ of an ordinary hydrogen atom are diagrammed, the result is as follows.

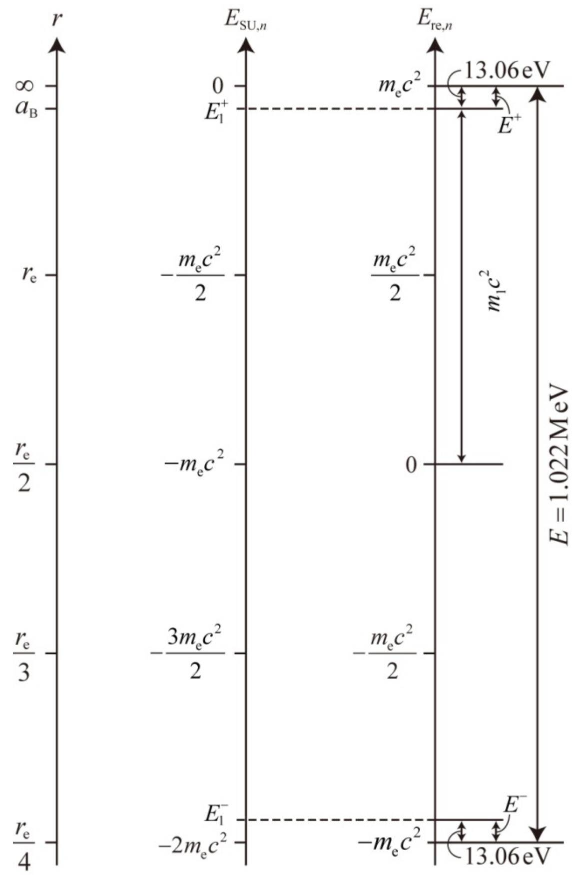

Fig. The original energy of an electron at rest is m_ec^2. However, when the energy levels of the hydrogen atom were described in quantum mechanics, the energy of an electron at rest was set to zero. Therefore, in the description at the level of classical quantum theory, the energy levels of the hydrogen atom are defined as follows.

$$E_{SU,n} = K_{re,n} + V(r_n) = -K_{re,n}$$

In contrast, the relativistic energy levels $E_{re,n}$ of an electron can be defined as follows.

$$E_{re,n} = m_nc^2 = m_ec^2 + K_{re,n} + V(r_n) = m_ec^2 - K_{re,n} = m_ec^2 + E_{SU,n}$$

An electron at the $E_{re,n}^-$ energy levels has negative mass, and exists near the atomic nucleus (proton). The author predicts that this unknown matter composed of a proton and an electron at ultra-low energy levels is the true nature of dark matter.

CLASSICAL ORBITAL RADII OF THE ELECTRON AT THE UNKNOWN ENERGY LEVELS

This section discusses the unknown orbital radii of a hydrogen atom.

Incidentally, the energy of the hydrogen atom can also be written as follows.

$$E_n = \frac{1}{2}V(r_n) = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_n} = -\frac{1}{2}\frac{r_em_ec^2}{r_n} = -m_ec^2\left(\frac{r_e/2}{r_n}\right). \quad (44)$$

Here, r_e is the classical electron radius as follows.

$$r_{\mathrm{e}} = \frac{e^2}{4\pi\varepsilon_0 m_{\mathrm{e}} c^2} = 2.8179403227 \times 10^{-15}\ \mathrm{m}. \tag{45}$$

Also, the following equation for energy can be obtained from Equation (44).

$$m_n c^2 = m_{\mathrm{e}} c^2 + E_n = m_{\mathrm{e}} c^2 \left(1 - \frac{r_{\mathrm{e}}/2}{r_n}\right). \tag{46}$$

Here, if $-m_{\mathrm{e}}c^2$ is substituted for *E* in Equation (44), then the *r* where $E_{\mathrm{re}} = 0$ is:

$$r = \frac{r_{\mathrm{e}}}{2}. \tag{47}$$

Dirac pointed out that there is a negative solution to Equation (9). Adopting the same viewpoint, there is a negative solution to Equation (37). To find the negative solution, it is necessary to create a quadratic equation for *r*. Thus, from Equations (42) and (46),

$$\left(\frac{r_n - r_{\mathrm{e}}/2}{r_n}\right)^2 = \frac{n^2}{n^2 + \alpha^2}. \tag{48}$$

From this, the following quadratic equation is obtained.

$$r_n^2 - \left(\frac{n^2 + \alpha^2}{\alpha^2}\right) r_{\mathrm{e}} r_n + \left(\frac{n^2 + \alpha^2}{\alpha^2}\right) \frac{r_{\mathrm{e}}^2}{4} = 0. \tag{49}$$

If this equation is solved for r_n,

$$r_n^{\pm} = \frac{r_{\mathrm{e}}}{2}\left(1 + \frac{n^2}{\alpha^2}\right)\left[1 \pm \left(1 + \frac{\alpha^2}{n^2}\right)^{-1/2}\right]. \tag{50}$$

When the Taylor expansion of Equation (50) is taken, the result is as follows.

$$r_n^{\pm} \approx \frac{r_{\mathrm{e}}}{2}\left(1 + \frac{n^2}{\alpha^2}\right)\left[1 \pm \left(1 - \frac{\alpha^2}{2n^2} + \frac{3\alpha^4}{8n^4}\right)\right]. \tag{51}$$

To begin, the positive solution is found first. (The positive solution is the solution found by Bohr.) The radii r_n^+ found from Equation (51) are as follows.

$$r_n^+ \approx \frac{3r_{\mathrm{e}}}{4} + \frac{r_{\mathrm{e}}}{\alpha^2} n^2 = \frac{3r_{\mathrm{e}}}{4} + a_{\mathrm{B}} n^2. \tag{52}$$

Here, a_{B} is the Bohr radius as follows.

$$a_{\mathrm{B}} = \frac{4\pi\varepsilon_0 \hbar^2}{m_{\mathrm{e}} e^2} = 0.52917721067 \times 10^{-10}\ \mathrm{m}. \tag{53}$$

In contrast, the radii r_n found by Bohr are given by the following equation.

$$r_n = a_{\mathrm{B}} n^2, \quad n = 1, 2, \cdots. \tag{54}$$

If Equations (52) and (54) are compared, it is evident that Equation (54) is an approximation.
Next, the negative solution r_n^- of Equation (51),

$$r_n^- \approx \frac{r_{\mathrm{e}}}{4} + \frac{\alpha^2 r_{\mathrm{e}}}{16n^2} = \frac{r_{\mathrm{e}}}{4} + \frac{a_{\mathrm{B}}}{n^2}\left(\frac{\alpha}{2}\right)^4. \tag{55}$$

Since r^- converges to $r_{\mathrm{e}}/4$, $r_{\mathrm{e}}/4$ can be regarded as the radius of the atomic nucleus of a hydrogen atom (i.e., the proton). Here, the theoretical value of the proton radius is:

$$\frac{r_{\mathrm{e}}}{4} = 0.704485080675 \times 10^{-15}\ \mathrm{m}. \tag{56}$$

However, if an attempt is actually made to measure the size of the proton (atomic nucleus), the energy of the proton changes. The size of the proton depends on the proton's energy, and thus the measured value does not match with Equation (56). In addition, it is possible to predict that a different measurement value will be obtained from an experiment using a different measurement method [27][28].

Incidentally, the orbital radii r_n of the electron is a classical concept. In quantum mechanics, r_n is defined in each stationary state as the radii where the probability that the electron is present is maximal. In this paper too, r_n is used in the sense of quantum mechanics.

The next compares the orbital radii of an electron in a hydrogen atom r_n^+ and the orbital radii of an electron with a negative mass r_n^-. Referring to Equation (50),

$$\frac{r_n^-}{r_n^+}=\left[1-\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}\right]\left[1+\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}\right]^{-1}=\frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left(n^2+\alpha^2\right)^{1/2}+n}. \tag{57}$$

Here, if we set $n=1$,

$$\frac{r_1^-}{r_1^+}=\frac{\left(1+\alpha^2\right)^{1/2}-1}{\left(1+\alpha^2\right)^{1/2}+1}=1.3312484168\times10^{-5}\approx\frac{1}{75120}. \tag{58}$$

Next, let's find $r_n^+ + r_n^-$ and $r_n^+ r_n^-$ from Equation (50). When this is done,

$$r_n^+ + r_n^- = r_{\mathrm{e}}\left(1+\frac{n^2}{\alpha^2}\right). \tag{59}$$

$$r_n^+ r_n^- = \frac{r_{\mathrm{e}}^2}{4}\left(1+\frac{n^2}{\alpha^2}\right). \tag{60}$$

From this,

$$\frac{r_n^+ r_n^-}{r_n^+ + r_n^-}=\frac{r_{\mathrm{e}}}{4}. \tag{61}$$

Next, if $\left(r_1^- - r_{\mathrm{e}}/4\right)$ and $r_{\mathrm{e}}/4$ are compared using r_1^- in Equation (55).

$$\frac{r_1^- - r_{\mathrm{e}}/4}{r_{\mathrm{e}}/4}\approx\frac{\alpha^2 r_{\mathrm{e}}}{16}\frac{4}{r_{\mathrm{e}}}=\frac{\alpha^2}{4}=1.331284168\times10^{-5}. \tag{62}$$

From this, it is evident that the electron with negative mass is located near the atomic nucleus. Incidentally, Equation (61) can be written as follows.

$$\frac{r_n^+ + r_n^-}{r_n^+}=\frac{r_n^-}{r_{\mathrm{e}}/4}. \tag{63}$$

This equation can be written as follows.

$$\frac{r_n^-}{r_n^+}=\frac{r_n^- - r_{\mathrm{e}}/4}{r_{\mathrm{e}}/4}. \tag{64}$$

Incidentally, the following equation can be derived from Equation (61).

$$r_n^+ = \frac{r_{\mathrm{e}}}{4}\frac{r_n^-}{r_n^- - r_{\mathrm{e}}/4}. \tag{65}$$

$$r_n^- = \frac{r_{\mathrm{e}}}{4}\frac{r_n^+}{r_n^+ - r_{\mathrm{e}}/4}. \tag{66}$$

From this, the following relationship is obtained from Equations (57) and (64).

$$\frac{r_n^-}{r_n^+}=\frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left(n^2+\alpha^2\right)^{1/2}+n}=\frac{r_n^- - r_{\mathrm{e}}/4}{r_{\mathrm{e}}/4}. \tag{67}$$

Using Equation (65), Equation (67) becomes the following.

$$\begin{aligned} r_n^+ &= \frac{r_{\mathrm{e}}}{4}\frac{r_n^+}{r_n^- - r_{\mathrm{e}}/4}\frac{r_n^-}{r_n^+} \\ &= \frac{r_{\mathrm{e}}}{4}\frac{r_n^+}{r_n^- - r_{\mathrm{e}}/4}\frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left(n^2+\alpha^2\right)^{1/2}+n}. \end{aligned} \tag{68}$$

Also, Equation (68) can be written as follows.

$$r_n^- - \frac{r_{\mathrm{e}}}{4}=\frac{r_{\mathrm{e}}}{4}\frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left(n^2+\alpha^2\right)^{1/2}+n}. \tag{69}$$

Next, if Equation (69) is solved for r_n^-,

$$\begin{aligned} r_n^- &= \frac{r_{\mathrm{e}}}{4}\left[1+\frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left(n^2+\alpha^2\right)^{1/2}+n}\right] \\ &= \frac{r_{\mathrm{e}}}{2}\frac{\left(n^2+\alpha^2\right)^{1/2}}{\left(n^2+\alpha^2\right)^{1/2}+n}. \end{aligned} \tag{70}$$

If r_n^+ is found from Equation (66) and Equation (67) using the same method,

$$r_n^+ = \frac{r_{\mathrm{e}}}{2}\frac{\left(n^2+\alpha^2\right)^{1/2}}{\left(n^2+\alpha^2\right)^{1/2}-n}. \tag{71}$$

Also, Equations (70) and (71) can be written as follows.

$$r_n^- = \frac{r_{\mathrm{e}}}{2}\left[1-\frac{n}{\left(n^2+\alpha^2\right)^{1/2}+n}\right]. \tag{72}$$

$$r_n^+ = \frac{r_{\mathrm{e}}}{2}\left[1+\frac{n}{\left(n^2+\alpha^2\right)^{1/2}-n}\right]. \tag{73}$$

Equations (72) and (73) have already been derived in another paper [23]. However, Equations (72) and (73) could not have been derived in Reference [23] without assuming Equation (67). However, in this paper, Equation (67) is derived theoretically, and thus Equations (72) and (73) are theoretically derived formulas.

The orbital radius of an electron is defined as the distance from the center of the atomic nucleus to the electron. However, according to Equation (72) and Equation (73), the sizes of r_n^- and r_n^+ are determined taking $r=r_{\mathrm{e}}/2$ as a starting point.

This is a discovery that has not been previously stated. r_n^- approaches the atomic nucleus (radius $r_{\mathrm{e}}/4$) as n increases. In quantum mechanics, lower energy is regarded as more stable, but that is incorrect. Actually, the closer the relativistic energy of an electron is to zero, the greater the stability. Therefore, electrons with negative energy are never easily incorporated into the atomic nucleus.

Incidentally, the probability that an electron exists at the energy levels $E_{\mathrm{re},n}^{-}$ is far higher than the probability that it exists at the energy levels $E_{\mathrm{re},n}^{+}$. Furthermore, a particle formed from a proton (atomic nucleus) and an electron at the energy levels $E_{\mathrm{re},n}^{-}$ is extremely small compared to an ordinary hydrogen atom.

The author uses the name "dark hydrogen atom" for matter formed from one proton with positive mass and one electron with negative mass.

Dark hydrogen formed when dark hydrogen atoms come together in mass, and the various types of particles formed from other dark atoms (dark matter) can easily take on a state of high density compared to ordinary matter.

Also, the wavelength of a photon emitted when an electron at the energy levels $E_{\mathrm{re},n}^{-}$ transitions matches the wavelength of a photon emitted from an ordinary hydrogen atom. Therefore, it is not possible to discriminate photons emitted from dark hydrogen atoms and those emitted from ordinary hydrogen atoms.

In this paper, the discussion focuses on the kinetic energy of an electron in a hydrogen atom, and the explanation of the electron's potential energy is inadequate. For a detailed explanation of potential energy, see Reference [6].

Also, the quantum number treated in this paper is the principal quantum number. The relativistic wave equation must be solved if a quantum-mechanical discussion including spin is needed. The author has previously derived a relativistic wave equation to take the place of the Dirac equation. If necessary, please refer to that paper [18][19].

Mathematically, there exist negative energy levels, and solutions of electron orbitals corresponding to them. This state becomes possible because the electron has latent negative energy. Details of this point have been described in Reference [25].

CONCLUSION

A previously unknown state exists, formed inside the hydrogen atom from an electron with negative mass and a proton (atomic nucleus) with positive mass (please see the separate paper for the experimental basis demonstrating the existence of an electron with negative mass [22][26]). The author predicts that particles in this unknown state are the true nature of dark matter. The thinking process leading to this prediction is summarized below.

1) The potential energy of the hydrogen atom corresponds to the decrease in rest mass energy of the electron. That is,

$$V(r) = -\Delta m_{\mathrm{e}} c^2. \tag{74}$$

2) Einstein's energy-momentum relationship cannot be applied to the electron in a hydrogen atom. The energy-momentum relationship applicable in the space inside the atom where potential energy exists is the following relationship.

$$\left(mc^2\right)^2 = -c^2 p^2 + \left(m_{\mathrm{e}} c^2\right)^2, \quad m < m_{\mathrm{e}}. \tag{75}$$

3) This relationship still does not incorporate the discontinuities of physical quantities characteristic of the micro world. To solve this problem, the author derived the following quantum condition to take the place of Bohr's quantum condition.

$$\frac{v_n}{c} = \frac{\alpha}{n}. \tag{76}$$

This was used to incorporate discontinuity into Equation (75).
4) As result, Equation (75) can be written as follows.

$$\left(m_n c^2\right)^2 + c^2 p_n^2 = \left(m_{\mathrm{e}} c^2\right)^2. \tag{77}$$

$$\left(m_{\mathrm{e}} c^2\right)^2 \left(\frac{n^2}{n^2+\alpha^2}\right) + c^2 \left(m_{\mathrm{e}} c\right)^2 \left(\frac{\alpha^2}{n^2+\alpha^2}\right) = \left(m_{\mathrm{e}} c^2\right)^2. \tag{78}$$

5) When Equation (77) is solved, it is evident that there are positive and negative solutions for the relativistic energy levels of the hydrogen atom. That is,

$$E_{\mathrm{re},n}^{\pm} = \pm m_n c^2 = \pm m_{\mathrm{e}} c^2 \mp K_{\mathrm{re},n} = \pm m_{\mathrm{e}} c^2 \left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}, \quad n = 0,1,2,\cdots. \tag{79}$$

Also, electrons at the $-m_n c^2$ energy levels have negative mass.
6) The orbital radius of the hydrogen atom is found by solving the following equation.

$$\left(\frac{r_n - r_{\mathrm{e}}/2}{r_n}\right)^2 = \frac{n^2}{n^2+\alpha^2}. \tag{80}$$

When this equation is solved, the orbital radius of an ordinary hydrogen atom r_n^+ is as follows.

$$r_n^+ = \frac{r_{\mathrm{e}}}{2}\left[1 + \frac{n}{\left(n^2+\alpha^2\right)^{1/2} - n}\right]. \tag{81}$$

In contrast, the orbitals r_n^- of electrons at ultra-low energy levels are

$$r_n^- = \frac{r_{\mathrm{e}}}{2}\left[1 - \frac{n}{\left(n^2+\alpha^2\right)^{1/2} + n}\right]. \tag{82}$$

Here, if we set $n=1$,

$$\frac{r_1^-}{r_1^+} = \frac{\left(1+\alpha^2\right)^{1/2} - 1}{\left(1+\alpha^2\right)^{1/2} + 1} = 1.3312484168 \times 10^{-5} \approx \frac{1}{75120}. \tag{83}$$

Next, if $\left(r_1^- - r_{\mathrm{e}}/4\right)$ and $r_{\mathrm{e}}/4$ are compared using r_1^- in Equation (55).

$$\frac{r_1^- - r_{\mathrm{e}}/4}{r_{\mathrm{e}}/4} \approx \frac{\alpha^2}{4} = 1.331284168 \times 10^{-5}. \tag{84}$$

An electron with negative mass exists near the atomic nucleus.
The following relation also holds due to Equations (57) and (65).

$$\frac{r_n^-}{r_n^+} = \frac{r_n^- - r_{\mathrm{e}}/4}{r_{\mathrm{e}}/4} = \frac{\left(n^2+\alpha^2\right)^{1/2} - n}{\left(n^2+\alpha^2\right)^{1/2} + n}. \tag{85}$$

The orbital radius of the electron is given as the distance from the center of the atomic nucleus (proton) to the electron. However, this paper has shown that the size of the electron orbital will be determined taking $r = r_e / 2$ as the starting point.

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