An energy-momentum relationship for a bound electron inside a hydrogen atom

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Abstract: Einstein's energy-momentum relationship, which is representative of the special theory of relativity, holds true in an isolated system in free space, but quantum mechanics as represented by the Dirac equation has been considered the best theory to describe the behavior of electrons possessing potential energy inside atoms. This paper asks the following question: "if a formula similar to Einstein's relationship, which holds true in free space, were to also hold true inside a hydrogen atom, what would such a formula look like?" and then derives this relationship. The newly derived formula of this paper could prove useful as a formula to supplement quantum mechanics. © *2011 Physics Essays Publication*. [DOI: 10.4006/1.3583810]

Résumé: L'équation d'Einstein reliant l'énergie et la quantité de mouvement, qui est représentative de la théorie de la relativité restreinte, est valide dans un système isolé de l'espace. Cependant, il est considéré en terme général que le comportement des électrons dans un atome est mieux décrit par les équations comme celle de Dirac, qui appartiennent au domaine de la physique quantique. Dans cet article, nous sommes partis de la question suivante: "Si une équation similaire à celle d'Einstein peut rester vrai pour un atome d'hydrogène, quelle sera alors la forme de cette nouvelle équation?" Ainsi, l'équation induite par ce travail pourrait s'avérer complémentaire à la mécanique quantique.

Key words: Einstein's Energy-Momentum Relationship; Special Theory of Relativity; Dirac Equation; Rest Mass Energy; Relativistic Energy; Hydrogen Atom; Potential Energy; Total Mechanical Energy.

I. INTRODUCTION

Generally, the physics theory that best describes the behavior of electrons inside atoms is thought to be quantum mechanics. This paper does not disagree with this theory but further poses the question of whether Einstein's energy-momentum relationship holds true even in the space inside atoms, and further attempts to answer to this question.

(In quantum mechanics, the motion of the electron inside the hydrogen atom cannot be conceived as a classical motion, but is really a standing wave ψ, so that $|\psi * \psi|$ is symmetric and time independent all around the proton. Hence, the charge distribution is perfectly static (no classical motion).

The above question should have been first posed and answered when Einstein announced the above mentioned relationship, that is, when quantum mechanics was still incomplete (around the 1920s). Based on this background, in this paper, we are going to derive the new relationship by using the approach of classical quantum theory.

II. CONFIRMING THAT EINSTEIN'S RELATIONSHIP DOES NOT HOLD TRUE INSIDE A HYDROGEN ATOM

The case of a single electron, at rest in free space, is considered. According to Einstein, the only energy of an electron in this state is its rest mass energy, or E_0, which is $m_e c^2$.[1]

If this electron absorbs photon energy, the photon energy absorbed is transferred into kinetic energy of the electron. We know that the following Einstein's energy-momentum relationship is true for the relativistic energy E and the momentum p of the electron that begins moving:

$$E^2 = c^2 p^2 + E_0^2. \tag{1}$$

Of course, $E > E_0$ in this case.

However, what happens if this electron at rest is attracted to the nucleus of a hydrogen atom, a proton, and is drawn into the atom?

The electron emits a photon from itself without absorbing external energy, and at the same time gains an amount of kinetic energy equivalent to the photon energy.

Referring to classical quantum theory, the relationship between the total mechanical energy and kinetic energy of an electron (K) inside a hydrogen atom is

$$E_n = -\frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{m_e e^4}{\hbar^2}\frac{1}{n^2} \tag{2a}$$

$$= \frac{E_1}{n^2} \tag{2b}$$

$$= -\frac{K_1}{n^2}, \quad n = 1, 2, \ldots . \tag{2c}$$

Here, n is the principal quantum number.

In this case, the total mechanical energy of the electron has a negative value (see Appendix A).

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According to classical quantum theory, the total mechanical energy of an electron is considered to be zero when the electron is separated from the atomic nucleus by a distance of infinity and remains at rest in that location. The total mechanical energy of Eq. (2a) is the value obtained from this perspective.

However, even if we place an electron at rest an infinite distance from its nucleus, the relativistic energy of the electron is fundamentally not zero. According to Einstein, an electron in this state should have rest mass energy.

According to quantum mechanics textbooks, the eigenvalue of the energy of a hydrogen atom as obtained from the Dirac equation, which is a relativistic wave equation, is as follows:[2]

$$E = m_e c^2 \left[1 - \frac{\gamma^2}{2n^2} - \frac{\gamma^4}{2n^4} \left(\frac{n}{|k|} - \frac{3}{4} \right) \right]. \tag{3}$$

It is important to note that energy here is defined on an absolute scale. Because $Z=1$ in the case of a hydrogen atom, $\gamma = e^2/\hbar c$, (γ is the fine structure constant). If we ignore for the third term of this equation and define it as an approximation, Eq. (3) can be written as follows:

$$E = m_e c^2 - \frac{1}{2} \left(\frac{1}{4\pi\varepsilon_0} \right)^2 \frac{m_e e^4}{\hbar^2} \frac{1}{n^2} \tag{4a}$$

$$= m_e c^2 + E_n. \tag{4b}$$

Moreover, E of Eq. (B4) defines an absolute quantity, which includes the electron's rest mass energy (see Appendix B).

Here, the energy of a hydrogen atom Eq. (2a) corresponds to the reduction in the electron's rest mass energy, while conversely, Eq. (4a) corresponds to the electron's remaining rest mass energy.

Even if the electron which was at rest begins moving in free space, and even if it is absorbed into an atom, the starting point of the electron's energy for either case is its rest mass energy.

The electron's relativistic energy E for the former state is $E > m_e c^2$ and for the latter state is $E < m_e c^2$. The above discussion has shown that Einstein's relationship (1) does not hold true inside a hydrogen atom. However, physicists do not normally consider this to be a problem.

Currently, Einstein's relationship (1) is used to describe the energy and momentum of particles in free space, but for explaining the behavior of electrons inside atoms, opinion has shifted to quantum mechanics as represented by equations such as the Dirac equation.

However, in this paper, while agreeing on the importance of the Dirac equation, from a perspective of symmetry, the following questions are posed: Would a formula similar to Einstein's relationship, which holds true in free space, also hold true in a hydrogen atom? If such a formula were to hold true, what would it look like?

When describing the motion of a bound electron inside a hydrogen atom, a term must be included in that formula for the potential energy.

Fortunately, in a separate paper, the author has asserted that the potential energy of a bound electron inside a hydrogen atom is equal to the reduction in the rest mass energy of that electron.[3]

Namely,

$$V(r) = -\Delta m_e c^2. \tag{5}$$

Let us consider this relationship as we derive the formula to describe the behavior of a bound electron inside a hydrogen atom in this paper.

III. RELATIONSHIP BETWEEN ENERGY AND MOMENTUM OF A BOUND ELECTRON INSIDE A HYDROGEN ATOM

Equation (1) is one of the most important relationships in the special theory of relativity (STR).

If we assume the particle in this case is an electron, an electron at rest in an isolated system will begin moving when it absorbs external energy.

Equation (6) shows the relationship between the electron's relativistic energy and its momentum and rest mass energy.

$$(mc^2)^2 = c^2 p^2 + (m_e c^2)^2. \tag{6}$$

The following equation is presumed to be true when deriving Eq. (1):[4]

$$dE = v\,dp. \tag{7}$$

When a particle moves through macroscopic space, for an isolated system, as its velocity increases, the kinetic energy and hence total mechanical energy of the particle will increase. In classical mechanics, the increase of kinetic energy corresponds to the work done by external forces, and we have

$$dK = F\,dx \tag{8a}$$

$$= \frac{dp}{dt} dx \tag{8b}$$

$$= v\,dp. \tag{8c}$$

Also, in this situation, the particle's total mechanical energy and kinetic energy increase, but the increases are equal. That is,

$$dE = dK. \tag{9}$$

Equation (7) can be subsequently derived from Eqs. (8c) and (9).

Next, let us imagine an electron that is at rest an infinite distance in macroscopic space from nucleus of a hydrogen atom—a proton—and is attracted by the proton's electrical force, creating a hydrogen atom. The electron emits photons outside the atom and reduces its rest mass energy, but at the same time gains an amount of kinetic energy equal to the reduced amount of energy.

The following relationship can be derived from Eq. (A5c).

$$-dE = dK. \tag{10}$$

The following relationship can be subsequently derived from Eqs. (8c) and (10).

$$-dE = v dp. \tag{11}$$

We start from Eq. (7) when deriving Einstein's relationship (1), but when deriving the relationship inside a hydrogen atom, we must start from Eq. (11).

Referring to a STR textbook, we derive the energy-momentum relationship for a bound electron inside a hydrogen atom.[5]

In classical mechanics,

$$m = \frac{p}{v}. \tag{12}$$

And, in STR,

$$m = \frac{E}{c^2}. \tag{13}$$

If, further, we suppose that Eq. (13) describes a universal equivalence of energy and inertial mass, we can combine Eqs. (12) and (13) into a single statement:

$$E = \frac{c^2 p}{v}. \tag{14}$$

Next, by multiplying the left and right sides of Eqs. (11) and (14), we obtain

$$EdE = -c^2 p dp. \tag{15}$$

We integrate this

$$E^2 = -c^2 p^2 + \text{const.} \tag{16}$$

We shall next determine the constant of integration and energy E for Eq. (16).

The constant of integration equation (16) should be normally determined through experimentation.

However, from the analogy of Eq. (1) of this discussion, the constant of integration Eq. (16) can be assumed to be $(m_e c^2)^2$.

Therefore, Eq. (16) becomes

$$E^2 + c^2 p^2 = (m_e c^2)^2. \tag{17}$$

Next, in order to derive the energy-momentum relationship established inside an atom, we must define the relativistic energy of the electron.

Incidentally, this paper may give the reader the impression that the hydrogen atom is simply a heavy proton and a small light electron orbiting around it.

Thus, in this paper, the relativistic energy, $E_{\mathrm{re},n}$ for a bound electron inside a hydrogen atom is defined as below.

$$E_{\mathrm{re},n} = m_e c^2 + K_n + V(r_n) \tag{18a}$$

$$= m_e c^2 + \frac{V(r_n)}{2} \tag{18b}$$

$$= m_e c^2 + E_n, \quad n = 1,2,\ldots, \quad E_n < 0. \tag{18c}$$

To agree with n on the left side, n is added to K and r on the right side. Here, $E_{\mathrm{re},n}$ is the relativistic energy when the principal quantum number is n. Also, E_n is the energy of a hydrogen atom in Bohr's solution, and we can clearly see that it is an approximate definition, as is shown by the Eq. (18c) ignoring the third term of Eq. (3).

In this paper, we use Eq. (18c) for E in Eq. (17). This enables us to rewrite Eq. (17) as

$$E_{\mathrm{re},n}^2 + c^2 p_n^2 = (m_e c^2)^2. \tag{19}$$

$$\left(m_e c^2 + \frac{V(r_n)}{2}\right)^2 + c^2 p_n^2 = (m_e c^2)^2. \tag{20}$$

$$(m_e c^2 + E_n)^2 + c^2 p_n^2 = (m_e c^2)^2, \quad n = 1,2,\ldots, \quad E_n < 0. \tag{21}$$

Equation (18c) is a nonrelativistic equation, although substituting this equation for one that is relativistic (17) raises doubts concerning the mixture of relativistic and nonrelativistic equations. However, Eq. (1) is normally considered a relativistic equation and can even actually be derived without some kind of relativistic request being required.

This is the most general equation that can be applied to particles moving at nonrelativistic speeds. However, when describing those moving at nonrelativistic speeds, since the approximation $E(v) \approx mc^2 + (1/2)mv^2$ is substituted, things add up even in the absence of Eq. (1).

Also, in the case of Eq. (19), the same logic is materialized. This equation represents the relationship between the energy and momentum of a bound electron in a system in which the energy levels are degenerating.

IV. RECALCULATING EXPRESSION (21)

We perform some tasks in this section to verify the accuracy of expression (21) derived in Sec. III.

First, we calculate the momentum p_n of an electron with an energy state having a principal quantum number n using classical quantum theory and the results of this paper. The following relationship exists between kinetic energy K_n and momentum p_n of an electron moving at a nonrelativistic speed and having an energy level with a principal quantum number n.

$$K_n \doteq \frac{p_n^2}{2m_e}. \tag{22}$$

By substituting the right side of Eq. (2a) for K_n of the above equation, we obtain the following:

$$\frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{m_e e^4}{\hbar^2}\frac{1}{n^2} \doteq \frac{p_n^2}{2m_e}. \tag{23}$$

By doing so, we obtain

$$p_n \doteq \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{m_e e^2}{n\hbar}. \tag{24}$$

We next derive p_n from Eq. (21). This can be rewritten as follows:

$$(m_e c^2 + E_n)^2 + c^2 p_n^2 = \left[m_e c^2 - \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{m_e e^4}{\hbar^2}\frac{1}{n^2}\right]^2 + c^2 p_n^2 \tag{25a}$$

$$= \left[m_e c^2 - \left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right)^2 \frac{m_e c^2}{2n^2}\right]^2 + c^2 p_n^2 \tag{25b}$$

$$= \left(1 - \frac{\alpha^2}{2n^2}\right)^2 (m_e c^2)^2 + c^2 p_n^2. \tag{25c}$$

By doing so, we obtain

$$\left(1 - \frac{\alpha^2}{2n^2}\right)^2 (m_e c^2)^2 + c^2 p_n^2 = (m_e c^2)^2. \tag{26}$$

However, α here is the following fine structure constant:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}. \tag{27}$$

By expanding Eq. (26), we obtain

$$p_n^2 = \left(\frac{\alpha^2}{n^2} - \frac{\alpha^4}{4n^4}\right)(m_e c)^2. \tag{28}$$

Incidentally, because $\alpha^4 = (5.325 \times 10^{-5})\alpha^2$, if we now set $\alpha^4/4n^4 \approx 0$, Eq. (28) can be written as

$$p_n \doteq \frac{\alpha m_e c}{n} \tag{29a}$$

$$\doteq \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{m_e e^2}{n\hbar}. \tag{29b}$$

As shown above, we find that p_n as derived from Eq. (21) is the same as the result derived from classical quantum theory (24). Thus, expression (21) has been shown to be true for a bound electron inside a hydrogen atom.

V. COMPARISON OF ENERGY OF A HYDROGEN ATOM AS DERIVED FROM THE DIRAC EQUATION AND FROM EXPRESSION (19)

In this section, we first use the method from quantum mechanics textbooks to derive the energy levels of a hydrogen atom using Dirac's relativistic wave equation.[6]

Then, we use expression (19) of this paper to derive the energy of a hydrogen atom, and then compare the two.

Dirac approached the problem of finding a relativistic wave equation by starting from the Hamiltonian form is as follows:

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = H\psi(\mathbf{r},t). \tag{30}$$

The simplest Hamiltonian that is linear in the momentum and mass term is as follows:

$$H = c\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m_e c^2. \tag{31}$$

By substituting this into Eq. (30), we obtain

$$(E - c\boldsymbol{\alpha}\cdot\mathbf{p} - \beta m_e c^2)\psi = 0. \tag{32}$$

We also quantize E and $\mathbf{p}$ from Eq. (32):

$$E \to i\hbar\frac{\partial}{\partial t}, \quad \mathbf{p} \to -i\hbar\nabla. \tag{33}$$

The result is the following quantized expression:

$$\left(i\hbar\frac{\partial}{\partial t} + i\hbar c\boldsymbol{\alpha}\cdot\nabla - \beta m_e c^2\right)\psi = 0. \tag{34}$$

Terms that involve the electromagnetic potentials can be added to Eq. (34) in a relativistic way by making the usual replacements as follows, where the particle described by the equation has electric charge e.

$$c\mathbf{p} \to c\mathbf{p} - e\mathbf{A}, \quad E \to E - e\varphi. \tag{35}$$

Here, E and $\mathbf{p}$ represent the operators of Eq. (33).

In special cases like a central field (in which $\mathbf{A}=0$ and φ are spherically symmetric), because $\mathbf{A}(\mathbf{r},t)=0$, $\varphi(\mathbf{r},t)=\varphi(r)$, this enables us to obtain

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi,$$

$$H = c\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m_e c^2 + V, \tag{36}$$

where $V = e\varphi$.

Represented in the same style as Eq. (34), we obtain

$$\left(i\hbar\frac{\partial}{\partial t} + i\hbar c\boldsymbol{\alpha}\cdot\nabla - \beta m_e c^2 - V\right)\psi = 0. \tag{37}$$

By performing the complex calculations based on this equation, we can obtain the following eigenvalue for the energy of a hydrogen atom (see Ref. 6 for detailed calculation).

$$E = m_e c^2\left[1 + \frac{\alpha^2}{(s+n')^2}\right]^{-1/2}, \quad n' = 0,1,2,\ldots. \tag{38}$$

Note: To follow the same notation as with other formulas, γ has been replaced with α.

Additionally, the correct quantum number for Eq. (38) is

$$s = \sqrt{k^2 - \alpha^2}, \quad n = n' + |k|. \tag{39}$$

The fine structure is made evident by expanding Eq. (38) in powers of α^2.

If we now substitute Eq. (39) into Eq. (38), and take the terms of order α^4, we obtain

$$E = m_e c^2\left[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\left(\frac{n}{|k|} - \frac{3}{4}\right)\right]. \tag{40}$$

n is the total quantum number, n' is the radial quantum number, and $|k|$ can take on positive integer values. Equation (40) shows that the energy increases with increasing $|k|$. This is the explanation as given in Schiff's textbooks.

TABLE I. Energy levels of a hydrogen atom as obtained from Eq. (48). Energy levels marked with ○ are the same as the energy as obtained from Eq. (44).

n	$n'(=n-\lvert k\rvert)$	$k[=\pm(n-n')]$	$l(=0,1,\cdots,n-1)$	$j(=\lvert l\pm 1/2\rvert)$	Energy levels	
1	0	1	0	1/2	$1S_{1/2}$	○
2	1	1	0	1/2	$2S_{1/2}$	
2	1	−1	1	1/2	$2P_{1/2}$	
2	0	2	1	3/2	$2P_{3/2}$	○
3	2	1	0	1/2	$3S_{1/2}$	
3	2	−1	1	1/2	$3P_{1/2}$	
3	1	2	1	3/2	$3P_{3/2}$	
3	1	−2	2	3/2	$3D_{3/2}$	
3	0	3	2	5/2	$3D_{5/2}$	○

The subscript "e" was not normally added to m in this book, but it has been added in this paper for consistency to differentiate between the rest mass of an electron and the relativistic rest mass in this paper.

When $n/|k|=1$ in Eq. (40), the contents of the third term in parentheses on the right side equal 1/4, so we can rewrite as

$$E_n = m_e c^2\left(1-\frac{\alpha^2}{2n^2}-\frac{\alpha^4}{8n^4}-\cdots\right). \tag{41}$$

Next, we shall derive the energy of a hydrogen atom from Eq. (19). First, we can rewrite Eq. (19) as

$$E_{\mathrm{re},n} = (m_e^2c^4 - c^2p_n^2)^{1/2} \tag{42a}$$

$$= m_e c^2\left(1-\frac{p_n^2}{m_e^2c^2}\right)^{1/2}. \tag{42b}$$

Then, using the relationship from Eq. (29a), an approximation, we obtain

$$E_{\mathrm{re},n} \doteq m_e c^2\left(1-\frac{\alpha^2}{n^2}\right)^{1/2}. \tag{43}$$

Using the binomial theorem expansion here, we obtain

$$E_{\mathrm{re},n} \doteq m_e c^2\left(1-\frac{\alpha^2}{2n^2}-\frac{\alpha^4}{8n^4}-\cdots\right). \tag{44}$$

The Dirac equation obtains the energy levels through complex calculations based on Eq. (37). However, we can easily obtain the energy levels by starting from Eq. (42b), which has been derived in this paper.

However, because Eq. (42b) is not a quantized expression, the energy value of Eq. (44) can only be obtained for the energy values of degenerative states.

VI. DISCUSSION

Because the solutions for the Dirac equation and for Eq. (43) are the same, we can predict that both expressions are equivalent under a certain condition.

That condition of equivalency is when n' of Eq. (38) is zero. When this is true, $n/|k|=1$, and the solutions for both expressions are the same, so the following equation must also be true:

$$\left(1+\frac{\alpha^2}{s^2}\right)^{-1} = 1-\frac{\alpha^2}{n^2}. \tag{45}$$

By substituting s from left side of this equation for Eq. (39), the left side of Eq. (45) becomes

$$\left(1+\frac{\alpha^2}{s^2}\right)^{-1} = \frac{k^2-\alpha^2}{k^2} \tag{46a}$$

$$= 1-\frac{\alpha^2}{n^2}. \tag{46b}$$

Ultimately, the energy values for Eqs. (38) and (43) are the same when $n'=0$ in Eq. (38). Incidentally, $|k|$ can be written as

$$|k| = j+\frac{1}{2}. \tag{47}$$

Thus, Eq. (40) can be written as

$$E = m_e c^2\left[1-\frac{\alpha^2}{2n^2}-\frac{\alpha^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}}-\frac{3}{4}\right)\right]. \tag{48}$$

$n/(j+1/2)-3/4$ is smallest here when $n/(j+1/2)=1$, and the energy level to satisfy this condition is the same as the energy level of Eq. (44). This energy level is the highest energy level of all available levels of all n. The corresponding energy levels are $1S_{1/2}(n=1,\ l=0,\ j=1/2)$, $2P_{3/2}(n=2,\ l=1,\ j=3/2)$, and $3D_{5/2}(n=3,\ l=2,\ j=5/2)$.

The levels with their nonrelativistic classifications are shown in the following table (see Table I).

From the above discussion, we can predict that the newly derived $E_{\mathrm{re},n}$ of this paper is equal to the Dirac equation energy eigenvalue (40). Thus, we obtain the following formula by replacing $E_{\mathrm{re},n}$ of Eq. (19) with E of Eq. (48):

$$E^2 + c^2p^2 = (m_e c^2)^2,$$

where

$$E = m_e c^2\left[1-\frac{\alpha^2}{2n^2}-\frac{\alpha^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}}-\frac{3}{4}\right)\right]. \tag{49}$$

VII. CONCLUSION

This paper has derived the energy-momentum relationship for a bound electron inside the hydrogen atom [Eq. (19)], an analog to Einstein's energy-momentum relationship for the free electron.

This paper posed the question of "what a formula would look like if Einstein's relationship, which holds true in free space, also held true inside a hydrogen atom, and what kind of formula would this be," and derived Eq. (19) as this formula.

$$E_{\mathrm{re},n}^2 + c^2 p_n^2 = E_0^2, \quad E_0 = m_e c^2.$$

It was easy to determine the energy of a degenerating hydrogen atom, or Eq. (44), from the newly derived Eq. (19) of this paper.

To derive this energy from the Dirac equation (37), however, complex calculations are required. Moreover, it required a great deal of effort to derive the Dirac equation from the Klein–Gordon equation obtained by quantizing Einstein's relationship (1).[7]

Based on the above, the newly derived formula of this paper is thought to be an important formula to supplement quantum mechanics. The issues of quantizing Eq. (19) will be the topic of another paper.

Although it was not touched on in this paper, it is known that the following two positive and negative values are obtained if Eq. (6) is solved for energy.

$$E = \pm \sqrt{c^2 p^2 + m_0^2 c^4}.$$

(This relation also holds for particles other than the electron, and thus m_e was changed to m_0 here.)

The following solution is obtained in the same way for Eq. (19):

$$E_{\mathrm{re},n} = \pm \sqrt{-c^2 p_n^2 + m_e^2 c^4}.$$

Oppenheimer once showed that the electron inside a hydrogen atom emits light and transitions to a negative energy state, and the atom disappears in an instant. This paper we will refrain from discussing this in detail, but the negative energy whose existence is predicted in this paper, $-\sqrt{-c^2 p_n^2 + m_e^2 c^4}$, can be regarded as the energy value when the electron inside the hydrogen atom has transitioned to a negative energy state.

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APPENDIX A

Let us review the energy of an electron inside a hydrogen atom. Let us suppose that an atomic nucleus is at rest because it is heavy, and consider the situation where an electron (electric charge $-e$, mass m_e) is orbiting at speed v along an orbit (radius r) with the atomic nucleus as its center.

An equation describing the motion is as follows:

$$\frac{m_e v^2}{r} = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r^2}. \tag{A1}$$

From this, we obtain:

$$\frac{1}{2} m_e v^2 = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r}. \tag{A2}$$

Meanwhile, the potential energy of the electron is

$$V(r) = -\frac{1}{4\pi\varepsilon_0} \frac{e^2}{r}. \tag{A3}$$

Since the right side of Eq. (A2) is $-1/2$ times the potential energy, Eq. (A2) indicates

$$V(r) = -2\left(\tfrac{1}{2} m_e v^2\right). \tag{A4}$$

Therefore, the total electron energy is

$$E = \tfrac{1}{2} m_e v^2 + V(r) \tag{A5a}$$

$$= -\frac{1}{2} m_e v^2 \tag{A5b}$$

$$= -K. \tag{A5c}$$

Also, the total energy of the electron is equal to half its potential energy,

$$E = \frac{1}{2} V(r). \tag{A6}$$

The reason for the difference in potential energy and kinetic energy in Eq. (A4) is thought to be the photonic energy $\hbar\omega$ released by the electron. Accordingly, we can establish the following law of energy conservation.

$$[V(r) + K] + \hbar\omega = 0. \tag{A7}$$

APPENDIX B

Gasiorowicz discussed the relativistic analog of Schrödinger for a bound (scalar) electron inside a hydrogen atom, which does include the rest mass energy of the electron in an attractive, central potential.[8]

This equation is

$$\left(\frac{E}{\hbar c} + \frac{Ze^2}{4\pi\varepsilon_0 \hbar c} \frac{1}{r}\right)^2 \psi = -\nabla^2 \psi + \left(\frac{m_e c}{\hbar}\right)^2 \psi, \tag{B1}$$

which is the operator version of Eq. (1) when a potential is included:

$$(E - V)^2 = c^2 p^2 + (m_e c^2)^2. \tag{B2}$$

The solution by solving for this Eq. (B1) did not agree with the actual energy level of the hydrogen atom. The reason proposed is that electrons are 1/2 spin particles and do not follow the Klein–Gordon equation. However, as a remaining problem, the left side of Eq. (B2) is as follows:

$$E - V = (K + V) - V \tag{B3a}$$

$$= K. \tag{B3b}$$

Thus, $K^2 > E_0^2$, or $(p^2/2m_e)^2 > (m_e c^2)^2$, but this kind of inequality should normally not be possible.

Here, let us surmise that E of Eq. (B2) is defined not as the E of Eq. (A5c) but instead as

$$E = m_e c^2 - K. \tag{B4}$$

By substituting this E into Eq. (B2) and considering the relation to Eq. (A4), we obtain

$$(m_e c^2 + K)^2 = c^2 p^2 + (m_e c^2)^2. \tag{B5}$$

This equation is identical to Einstein's relationship. In the end, the total energy E of Eq. (B2) is the energy as defined by Eq. (B4). E of Eq. (B2) includes the electron's rest mass energy and is defined on an absolute scale.

[1]A. Einstein, Technical Journal **5**, 16 (1946).
[2]L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), p. 486.
[3]K. Suto, Phys. Essays **22**, 135 (2009); **22**, 359 (2009).
[4]A. P. French, *Special Relativity* (W.W. Norton & Co., New York, 1968), p. 21.
[5]A. P. French, *Special Relativity* (in Ref. 4), p. 20.
[6]L. I. Schiff, *Quantum Mechanics* (in Ref. 2), p. 472.
[7]P. A. M. Dirac, *Directions in Physics* (Wiley, New York, 1978).
[8]S. Gasiorowicz, *Quantum Physics* (Wiley International Edition, New York, 2003), p. 144.