# Theoretical prediction of the size of the proton and correction of the quantum condition

Koshun Suto[a)]
*5-24, Oote-Town, Isesaki-City, 372-0048, Japan*

(Received 19 October 2011; accepted 24 July 2012; published online 12 October 2012)

**Abstract:** This paper shows that the quantum condition that Bohr applied to the hydrogen atom is an approximation. Also, as a result of calculations using a quantum condition newly assumed in this paper, it was possible to obtain, in addition to the orbital radius derived by Bohr, a physical quantity $r_e/4$, which is thought to be the radius of the proton. The electron is thought to be an elementary particle that has no size, but if it is assumed that the electron has a size, then this paper presents a value that is a strong candidate. If it is assumed that because the mass of the electron is involved in the size of the proton, the mass of the proton is involved in the size of the electron, then it becomes possible to predict the radius of the electron. If it is assumed that the radii of the proton and electron have existed in a fixed state since before the measurement of their size by experiment, then it becomes necessary to review quantum mechanics. © *2012 Physics Essays Publication*. [DOI: 10.4006/0836-1398-25.4.488]

**Résumé:** L'on montre ici que les conditions quantiques que Bohr appliquait à l'atome d'hydrogène sont une approximation. Aussi, d'après le résultat des calculs utilisant les conditions quantiques nouvellement supportées dans cet article, il est possible d'obtenir, en addition du rayon orbital dérivé de Bohr, une quantité physique $r_e/4$, qui est vraisemblablement le rayon du proton. On suppose que l'électron soit une particule élémentaire qui n'a pas de taille, mais si l'on considère que l'électron a une taille, alors cet article présente une valeur qui est une forte candidate. Si l'on suppose que, puisque la masse de l'électron est impliquée dans la taille du proton, la masse du proton est impliquée dans la taille de l'électron, alors il devient possible de prédire le rayon de l'électron. Si l'on suppose que les rayons du proton et de l'électron ont existé dans un état fixe jusqu'à la mesure de leur taille par expérimentation, il devient alors nécessaire de revoir les dynamiques quantiques.

Key words: Quantum Condition; Size of the Proton; Einstein's Energy-Momentum Relationship; Special Theory of Relativity; Rest Mass Energy; Classical Quantum Theory; Relativistic Energy; Hydrogen Atom; Potential Energy; Classical Electron Radius.

## I. INTRODUCTION

In classical quantum theory, the hydrogen atom is explained using a model in which an electron with negative charge rotates around a proton with positive charge due to the Coulomb attraction. If the atomic nucleus is assumed to be at rest because it is heavy, then the electron (charge $e$, mass $m_e$) is regarded as rotating at a speed $v$ along a circular orbit with radius $r$, centered on the nucleus. The total mechanical energy of the moving electron can be found from the sum of its kinetic energy and potential energy. The attraction that the electron receives from the proton is a central force, and the equation of motion can expressed as follows:

$$\frac{m_e v^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}. \tag{1}$$

Therefore,

$$\frac{1}{2}m_e v^2 = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}. \tag{2}$$

In addition, the potential energy $V(r)$ of the electron is assumed to be zero when the electron is at rest at a position infinitely far from the proton, and thus it becomes smaller than that inside the atom and can be described as follows:

$$V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}. \tag{3}$$

Because the right side of Eq. (2) is –1/2 times Eq. (3),

$$\frac{1}{2}m_e v^2 = -\frac{1}{2}V(r). \tag{4}$$

Finally, the total mechanical energy $E$ of the electron is

$$E = \frac{1}{2}m_e v^2 + V(r) = -\frac{1}{2}m_e v^2. \tag{5}$$

Also, if this energy is expressed using potential energy, then

$$E = \frac{1}{2}V(r) = -\frac{1}{8\pi\varepsilon_0}\frac{e^2}{r}. \tag{6}$$

To explain that the energy of the electron inside the atom assumes discrete states, Bohr thought in the following way:

a)koshun_suto129@mbr.nifty.com

 

If an integral number of electron waves of wavelength $\lambda$ is assumed to exist in the circumference $2\pi r$,

$$2\pi r = n\lambda, \tag{7}$$

and the de Broglie relationship $\lambda = h/p$ is substituted for $\lambda$ in this equation, then

$$p_n \cdot 2\pi r = 2\pi n\hbar, \qquad n = 1, 2, \ldots . \tag{8}$$

This is the quantum condition that Bohr assumed.

Next, if both sides of Eq. (2) are multiplied by $r^2$ and Eq. (8) is used, then

$$r_n = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} \cdot n^2 = a_B n^2, \qquad n = 1, 2, \ldots . \tag{9}$$

Here, the subscript $n$ is affixed to $r$ on the left side. $a_B$ is the Bohr radius, and $n$ is the principal quantum number.

Next, if Eq. (9) is substituted for $r_n$ in Eq. (6) and the subscript $n$ is affixed to $E$, too,

$$E_n = -\frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{m_e e^4}{\hbar^2} \cdot \frac{1}{n^2} = \frac{E_1}{n^2}, \qquad n = 1, 2, \ldots . \tag{10}$$

Above are the electron orbital radius and stationary state energy inside the hydrogen atom, derived from classical quantum theory. However, if a more rigorous calculation is necessary, $m_e$ must be replaced with the following reduced mass:

$$\mu = \frac{m_e m_p}{m_e + m_p}. \tag{11}$$

Incidentally, the reason Eq. (8) was used as the quantum condition in classical quantum theory is that the stationary state energy of the hydrogen atom can be correctly derived by using it. There were no better grounds than that. Therefore, Bohr did not prove that the two sides of Eq. (8) are equal. He simply assumed that they were equal.

If an electron that was at rest in free space is taken into the atom, the electron will gain kinetic energy, but at the same time, it will emit a photon with the same amount of energy. These energies are provided by a reduction in the potential energy of the electron. In a separate paper, the author has shown that the potential energy of the electron corresponds to the amount of reduction in rest mass energy of the electron.[1]

If the amount of this reduction is expressed as $-\Delta m_e c^2$, then

$$\begin{aligned} &-\Delta m_e c^2 = V(r_n) = 2E_n = -(K_n + \hbar\omega), \\ &K_n = \hbar\omega. \end{aligned} \tag{12}$$

(If multiple photons are emitted here, then $\hbar\omega$ is assumed to express the total sum of the energies of the individual photons. Also, kinetic energy is changed to $K$.) This shows that the following inequalities hold for the kinetic energy and potential energy of the electron inside the hydrogen atom.

$$K \leq \frac{1}{2} m_e c^2. \tag{13}$$

$$-m_e c^2 \leq V(r). \tag{14}$$

Incidentally, in quantum mechanics, the motion of the electron inside the hydrogen atom cannot be conceived as a classical motion but is really a standing wave $\psi$ so that $|\psi^*\psi|$ is symmetric and time independent all around the proton. Hence, the charge distribution is perfectly static (no classical motion). However, the argument in this paper should have continued after Bohr announced the classical quantum theory. Therefore, in this paper, it is regarded as permissible to argue using the classical quantum theory picture, without using complete quantum mechanics.

## II. ENERGY AND MOMENTUM OF AN ELECTRON INSIDE THE HYDROGEN ATOM

The author has shown, in a separate paper that the following relation holds between the energy and momentum of an electron inside a hydrogen atom.[2]

$$E_{re,n}^2 + c^2 p_n^2 = (m_e c^2)^2, \qquad n = 1, 2, \ldots . \tag{15}$$

Here, in this paper, the relativistic energy, $E_{re,n}$, for a bound electron inside a hydrogen atom is defined as below.

$$E_{re,n} = m_e c^2 + K_n + V(r_n) \tag{16a}$$

$$= m_e c^2 - \frac{1}{2}\Delta m_e c^2 \tag{16b}$$

$$= m_e c^2 + E_n. \tag{16c}$$

If an electron is at rest at a position infinitely far from a proton, the potential energy of that electron is zero. However, even in that case, the electron has rest mass energy of $m_e c^2$, and thus the exact energy of the electron is given not by Eq. (10) but by Eq. (16c). From this, $E_n$ can be defined as follows:

$$E_n = -(m_e c^2 - E_{re,n}). \tag{17}$$

Now, if Eq. (16c) is substituted for Eq. (15),

$$(m_e c^2 + E_n)^2 + c^2 p_n^2 = (m_e c^2)^2. \tag{18}$$

Next, if the value of Eq. (10) is substituted for $E_n$ in Eq. (18), then the left side of Eq. (18) becomes as follows:

$$\left[m_e c^2 - \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{m_e e^4}{\hbar^2} \cdot \frac{1}{n^2}\right]^2 + c^2 p_n^2$$

$$= \left[m_e c^2 - \left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right)^2 \frac{m_e c^2}{2n^2}\right]^2 + c^2 p_n^2 \tag{19a}$$

$$= (m_e c^2)^2 \left(1 - \frac{\alpha^2}{2n^2}\right)^2 + c^2 p_n^2. \tag{19b}$$

From this, Eq. (18) becomes

$$(m_e c^2)^2\left(1-\frac{\alpha^2}{2n^2}\right)^2 + c^2 p_n^2 = (m_e c^2)^2. \tag{20}$$

Here, $\alpha$ is the fine-structure constant defined as follows:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}. \tag{21}$$

The following physical quantities can also be found from Eq. (20):

$$E_{\mathrm{re},n} = m_e c^2\left(1-\frac{\alpha^2}{2n^2}\right), \tag{22}$$

$$E_n = -\frac{1}{2}m_e c^2\left(\frac{\alpha}{n}\right)^2, \tag{23}$$

where

$$p_n = m_e c\sqrt{\frac{\alpha^2}{n^2}-\frac{\alpha^4}{4n^4}} \tag{24a}$$

$$= m_e c\left(\frac{\alpha}{n}\right)\sqrt{1-\frac{\alpha^2}{4n^2}}. \tag{24b}$$

Incidentally, because $\alpha^4 = 2.836\times10^{-9}$, if we now set $\alpha^4/4n^4 \approx 0$, Eq. (24a) can be written as

$$p_n \approx \frac{\alpha m_e c}{n}. \tag{25}$$

This $p_n$ can also be found using another method. The following relationship exists between kinetic energy $K_n$ and momentum $p_n$ of an electron moving at a nonrelativistic speed and having an energy level with a principal quantum number $n$.

$$K_n \approx \frac{p_n^2}{2m_e}. \tag{26}$$

By substituting the right side of Eq. (10) for $K_n$ of the above equation, we obtain the following:

$$\frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{m_e e^4}{\hbar^2}\cdot\frac{1}{n^2}\approx\frac{p_n^2}{2m_e}. \tag{27}$$

By doing so, we obtain

$$p_n \approx \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{m_e e^2}{\hbar}\cdot\frac{1}{n} = \frac{\alpha m_e c}{n}. \tag{28}$$

Next, we consider the energy of the electron inside the hydrogen atom by referring to the logic given in textbooks.[3] If the velocity of the electron is set at zero in Eq. (15), then the following equation of Einstein can be derived:

$$m_e = \frac{E_{\mathrm{re}}}{c^2}. \tag{29}$$

Also, in classical mechanics,

$$m_e = \frac{p}{v}. \tag{30}$$

From these two equations, we obtain

$$cp = \frac{E_{\mathrm{re}} v}{c}. \tag{31}$$

The physical quantities of the electron inside the hydrogen atom take discrete values, and thus if the subscript $n$ is attached to the physical quantities on both sides of Eq. (31), then

$$cp_n = \frac{E_{\mathrm{re},n} v_n}{c}. \tag{32}$$

Substituting $cp_n$ in Eq. (32) for Eq. (15) here and simplifying and using the + value, we obtain

$$E_{\mathrm{re},n} = \frac{m_e c^2}{\sqrt{1+v_n^2/c^2}}. \tag{33}$$

Also taking into account the relationship in Eq. (29),

$$m_{\mathrm{re},n} = \frac{m_e}{\sqrt{1+v_n^2/c^2}}. \tag{34}$$

Here, the following $\gamma_{\mathrm{a},n}$ is defined by taking into account Eq. (22) and Eq. (33):

$$\gamma_{\mathrm{a},n} = \frac{1}{\sqrt{1+v_n^2/c^2}} = 1-\frac{\alpha^2}{2n^2} = 1+\frac{E_n}{m_e c^2}. \tag{35}$$

(The constant of proportionality in the Lorentz transformation is borrowed for $\gamma$. Because this constant of proportionality holds inside the atom, the "a" from "atom" is used as a subscript.)

When this is done, Eq. (33) and Eq. (34) can be expressed as follows:

$$E_{\mathrm{re},n} = \gamma_{\mathrm{a},n} m_e c^2 = m_e c^2\left(1-\frac{\alpha^2}{2n^2}\right). \tag{36}$$

$$m_{\mathrm{re},n} = \gamma_{\mathrm{a},n} m_e = m_e\left(1-\frac{\alpha^2}{2n^2}\right). \tag{37}$$

If the velocity of the electron inside the atom increases, then the energy and mass of the electron decrease. In addition, it can be predicted that the velocity of light does not function as a limit velocity inside the atom.

Incidentally, the relativistic energy in Eq. (33) becomes as follows for the extremely small $v_n/c$:

$$E_{\mathrm{re},n} = m_e c^2\left(1+\frac{v_n^2}{c^2}\right)^{-1/2} \approx m_e c^2\left(1-\frac{1}{2}\frac{v_n^2}{c^2}+\frac{3}{8}\frac{v_n^4}{c^4}+\cdots\right). \tag{38}$$

If $v_n/c$ is sufficiently small, this series can be approximated with high precision by the first two terms. That is,

$$E_{\mathrm{re},n} \approx m_e c^2\left(1-\frac{1}{2}\frac{v_n^2}{c^2}\right) = m_e c^2 - \frac{1}{2}m_e v_n^2. \tag{39}$$

Next, let us consider $r_n$. If the subscript $n$ is attached to $E$ and $r$ in Eq. (6),

$$E_n = -\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 r_n}. \tag{40}$$

If this expression is combined with Eq. (23) using an equal sign,

$$-\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 r_n} = -\frac{1}{2}m_e c^2\left(\frac{\alpha}{n}\right)^2. \tag{41}$$

Then,

$$r_n = \frac{1}{4\pi\varepsilon_0}\frac{e^2 n^2}{m_e c^2 \alpha^2} \tag{42a}$$

$$= a_B n^2 \tag{42b}$$

$$= r_e\left(\frac{n}{\alpha}\right)^2. \tag{42c}$$

Here, $r_e$ is the classical electron radius as follows:

$$r_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{m_e c^2}. \tag{43}$$

## III. PRESENTATION OF A NEW QUANTUM CONDITION AND THEORETICAL PREDICTION OF THE SIZE OF THE PROTON

When Bohr conjectured the quantum condition Eq. (8), he regarded the wavelength, which appears in Eq. (7), to be the same as the wavelength of the waves that appears in de Broglie's equation. However, according to quantum mechanics, there is no actual reality to the waves inside the atom. Bohr's quantum condition appeared as a hypothesis whose validity could not be logically proved.

In this paper, the orbital radius of the electron inside the hydrogen atom was obtained in Eq. (42c), and the formula for the momentum of the electron was obtained in Eq. (24b). Therefore, let us substitute these values into the left side of Eq. (8) and confirm whether or not this quantum condition actually holds.

First, the left side of Eq. (8) is

$$p_n \cdot 2\pi r_n = 2\pi\frac{n^2 r_e}{\alpha^2}\frac{\alpha m_e c}{n^2}\sqrt{n^2 - \alpha^2/4} \tag{44a}$$

$$= 2\pi\frac{r_e m_e c}{\alpha}\sqrt{n^2 - \alpha^2/4}. \tag{44b}$$

Here,

$$\frac{r_e m_e c}{\alpha} = \frac{e^2}{4\pi\varepsilon_0 m_e c^2}\cdot m_e c\left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right)^{-1} = \hbar. \tag{45}$$

Therefore,

$$p_n \cdot 2\pi r_n = 2\pi\hbar\sqrt{n^2 - \alpha^2/4}. \tag{46}$$

As is clear from Eq. (24b), regarding $\alpha^2/4 \approx 0$ means that the approximation value $\alpha m_e c/n$ is used as the value $p_n$:

$$p_n \cdot 2\pi r_n \approx 2\pi\frac{n^2 r_e}{\alpha^2}\frac{\alpha m_e c}{n} \tag{47a}$$

$$= 2\pi n\hbar. \tag{47b}$$

Due to the above considerations, it was possible to prove that the quantum condition Eq. (8) is an approximation.

Incidentally, a more generalized quantum condition is given not by Eq. (8) but by the following equation:

$$\oint p ds = 2\pi n\hbar. \tag{48}$$

Equation (46) is not what was presented as a quantum condition to replace Eq. (48).

Next, let us discuss whether the new quantum condition Eq. (46) can contribute to the development of physics. First, from Eq. (18), we find $p_n$ as follows:

$$p_n = \frac{1}{c}\sqrt{-2m_e c^2 E_n - E_n^2}. \tag{49}$$

Equation (46) can also be expressed as

$$p_n = \frac{\hbar\sqrt{n^2 - \alpha^2/4}}{r_n}. \tag{50}$$

Substituting the value of Eq. (50) for momentum in Eq. (49), we obtain

$$r_n\sqrt{-2m_e c^2 E_n - E_n^2} = \hbar c\sqrt{n^2 - \alpha^2/4}. \tag{51}$$

Here, substituting the right side of Eq. (40) for $E_n$, we obtain

$$\left[-2m_e c^2\left(-\frac{1}{2}\right)\frac{e^2}{4\pi\varepsilon_0 r_n} - \left(-\frac{1}{2}\right)^2\left(\frac{e^2}{4\pi\varepsilon_0 r_n}\right)^2\right] r_n^2$$
$$= \hbar^2 c^2\left(n^2 - \frac{\alpha^2}{4}\right). \tag{52}$$

Solving for $r_n$ in this equation, we obtain the following value:

$$r_n = \frac{1}{4}\frac{e^2}{4\pi\varepsilon_0 m_e c^2} + \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2}\cdot n^2 - \frac{\pi\varepsilon_0\hbar^2\alpha^2}{m_e e^2} \tag{53a}$$

$$= \frac{1}{4}\frac{e^2}{4\pi\varepsilon_0 m_e c^2} + \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2}\cdot n^2 - \frac{\pi\varepsilon_0\hbar^2}{m_e e^2}\left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)^2 \tag{53b}$$

$$= \frac{r_e}{4} + a_B n^2 - \frac{r_e}{4} \tag{53c}$$

$$= a_B n^2. \tag{53d}$$

In addition, Eq. (53c) can also be expressed as follows:

$$r_n = \frac{r_e}{4} + \frac{r_e n^2}{\alpha^2}\left(1 - \frac{\alpha^2}{4n^2}\right) = r_e\left(\frac{n}{\alpha}\right)^2,$$
$$n = 1, 2, \ldots. \tag{54}$$

Equations (9) and (53d) are mathematically the same formula, but they differ in what they signify. In the classical quantum theory, the proton is treated as a material particle, and thus it is not possible to discuss the distance from the surface of a proton that has size to the orbital.

However, if Eq. (53d) is expressed like Eq. (54), then the difference from Eq. (9) becomes clear. In Eq. (54), the first term is the radius of the proton, and $r_e n^2/\alpha^2$ can be interpreted as the distance from the surface of the proton to the orbital of the electron (see Appendix). According to this interpretation, the radius of the proton $r_p$ becomes 0.705 fm (where 1 fm $= 10^{-15}$ m). The value recommended by the Committee on Data for Science and Technology is 0.8768(69) fm, but recently experimental results have been reported indicating that the proton is a little smaller.[4] Also, values of 1.2 to 1.5 fm have been used until recently in physics textbooks, and we feel it is possible to regard the first term in Eq. (53c) as the radius of the proton.

Now, let us consider the first two terms of Eq. (53c). The proton radius and orbital radius are both distances measured from the center of the atomic nucleus. Therefore, these are distances in coordinate space, not in physical space, and up to $r_e/4$ from the center of the proton, these two radii overlap. The third term in Eq. (53c) signifies the operation of subtracting this overlap. If the value $\alpha m_e c/n$ for $p_n$ obtained from the previous quantum condition in Eq. (8) is substituted into Eq. (49), the following radius is obtained:

$$r_n = \frac{r_e}{4} + a_B n^2. \tag{55}$$

The orbital of the hydrogen atom in this case becomes larger than Eq. (9) by just the amount that was added to the proton radius. However, the equation for energy, Eq. (10), is not obtained even if the value of Eq. (55) is substituted for $r_n$ in Eq. (40).

Generally it is thought that $r_n$ and $E_n$ are dependent on $a_B$ and $E_1$, as is also evident from Eqs. (9) and (10). However, in this paper, Eqs. (54) and (23) are taken into account, and it is predicted that $r_n$ and $E_n$ are related to $r_e$ and $m_e c^2/2$. Incidentally, the physical quantities that determine the size of a proton are the electrical charge $e$ and the electron's rest mass energy $m_e c^2$.

Referring to the idea that de Broglie used when he predicted the existence of the matter wave, it becomes possible to discuss the size of an electron, which is considered a particle without extent.

In this paper, we have found that the mass of an electron $m_e$ is concerned in the size of a proton. Supposing the mass of a proton $m_p$ is concerned in the size of an electron, the radius of an electron $r_{el}$ is as follows:

$$r_{el} = \frac{1}{4}\frac{e^2}{4\pi\varepsilon_0 m_p c^2} = \frac{m_e r_p}{m_p} = 3.84 \times 10^{-19}\ \text{m}. \tag{56}$$

If it is assumed that the radii of the proton and electron have existed in a fixed state since before the measurement of their size by experiment, then it becomes necessary to review quantum mechanics.

According to the Copenhagen interpretation regarded as the conventional one on quantum mechanics, the microscopic particle, i.e., quantum, "behaves like a wave until its position is observed. But the moment its position is observed, its position as a particle is defined."

However, we have obtained the size of a proton by calculation, not by experiment. This means that a proton, a kind of quantum, is localized in a certain place as a particle, even if the position is not defined by observation. By the way, in the famous two-slit interference experiment with an electron, the conventional interpretation is that "an indivisible electron behaves as if it had come through both slits simultaneously."

However, this paper concludes as "although an electron comes through either slit as a particle, the probability distribution of electrons found by the detector draws a pattern of interference in the end."

If the prediction of this paper is correct, the Copenhagen interpretation ought to be revised.

## IV. CONCLUSION

(1) In Bohr's theory of the atom, the quantum condition is for selecting the stationary state assumed for electrons inside the hydrogen atom from the equations of motion in classical mechanics, and Eq. (8) has previously been regarded as correct. However, this paper has shown that Eq. (8) is nothing more than an approximation, and that more accurately, the following relationship holds:

$$\begin{aligned}\oint p ds &= 2\pi n\hbar\sqrt{1-\frac{\alpha^2}{4n^2}} = 2\pi n\hbar\sqrt{\frac{r_n - r_e/4}{r_n}} \\ &= 2\pi n\hbar\sqrt{1+\frac{1}{4}\frac{V(r_n)}{m_e c^2}}.\end{aligned} \tag{57}$$

The reason the quantum condition in Eq. (8) became an approximation is that in classical quantum theory, the $p_n$ in Eq. (25), which is an approximate value, was used instead of the $p_n$ in Eq. (24).

Finally, when Eqs. (8) and (57) are compared, it can be concluded that Bohr's quantum condition, Eq. (8), is the quantum condition that holds when the nucleus of the hydrogen atom is regarded as a point, and Eq. (57) proposed by the author is the quantum condition that holds when the size of the nucleus is taken into account.

(2) It was possible to find the orbital radius of the hydrogen atom to replace Eq. (9) by applying the quantum condition newly presented in this paper to Eq. (18):

$$\begin{aligned}&r_n = a_B n^2 = \frac{r_e}{4} + \frac{r_e n^2}{\alpha^2}\left(1-\frac{\alpha^2}{4n^2}\right) = r_e\left(\frac{n}{\alpha}\right)^2, \\ &n = 1, 2, \ldots .\end{aligned} \tag{58}$$

The term $r_e/4$, which is newly added to Eq. (54), is predominantly considered to be related to the atomic nucleus, or in other words, the proton radius. The radius of the proton $r_p$ is as follows:

$$r_p = \frac{r_e}{4} = 0.705\times 10^{-15}\ \text{m} = 0.705\ \text{fm}. \tag{59}$$

If Eq. (15) had been discovered in the era when Bohr developed classical quantum theory, then Bohr, too, would likely have derived Eq. (58) rather than Eq. (9) as the orbital radius of the hydrogen atom.

However, as is clear when Eq. (9) and Eq. (58) are compared, the distance to each orbital from the center of the nucleus is the same in both equations, and thus the energy values derived from Eq. (58) perfectly match the values derived by Bohr Eq. (10).

This paper has shown that the size of the nucleus of the hydrogen atom can be derived based on classical considerations predating the advent of quantum mechanics. However, that does not mean that this paper claims, based on Eq. (58), that the electron in the hydrogen atom moves in a classical orbital. In addition, the author would like to emphasize that even though this paper does not use quantum mechanics, it is not being claimed that atomic structure can be accounted for based just on classical theoretical argument.

## APPENDIX

This appendix provides a more detailed explanation.

First, the relativistic mass of an electron $m_{\mathrm{re},n}$ is defined as follows by referring to Eq. (16c):

$$m_{\mathrm{re},n} = \frac{E_{\mathrm{re},n}}{c^2} = \frac{1}{c^2}(m_{\mathrm{e}}c^2 + E_n). \tag{A1}$$

If this is used to find $p_n^2$ from Eq. (18), the result is

$$p_n^2 = (m_{\mathrm{e}}^2 - m_{\mathrm{re},n}^2)c^2. \tag{A2}$$

In addition, $K_n$ can be defined using the following equation:

$$K_n = -E_n = (m_{\mathrm{e}} - m_{\mathrm{re},n})c^2. \tag{A3}$$

The following relationship can be derived from these two equations:

$$K_n = \frac{p_n^2}{m_{\mathrm{e}} + m_{\mathrm{re},n}}. \tag{A4}$$

However, the following relationship holds in classical dynamics:

$$K = \frac{p^2}{2m} = \frac{1}{2}mv^2. \tag{A5}$$

Here, let us explain the reason why Eqs. (A4) and (A5) are different.

First, the following equation can be derived from Eq. (24):

$$p_n^2 = (m_{\mathrm{e}}c)^2\left(\frac{\alpha^2}{n^2} - \frac{\alpha^4}{4n^4}\right) \tag{A6a}$$

$$= \left(\frac{\alpha m_{\mathrm{e}}c}{n}\right)^2\left(1 - \frac{\alpha^2}{4n^2}\right). \tag{A6b}$$

Equation (37) can be rewritten as follows:

$$m_{\mathrm{e}} + m_{\mathrm{re},n} = 2m_{\mathrm{e}} - \frac{\alpha^2 m_{\mathrm{e}}}{2n^2} = 2m_{\mathrm{e}}\left(1 - \frac{\alpha^2}{4n^2}\right). \tag{A7}$$

Taking into consideration Eqs. (A6b) and (A7), the right side of Eq. (A4) is as follows:

$$\frac{p_n^2}{m_{\mathrm{e}} + m_{\mathrm{re},n}} = \frac{(\alpha m_{\mathrm{e}}c/n)^2(1 - \alpha^2/4n^2)}{2m_{\mathrm{e}}(1 - \alpha^2/4n^2)} = \frac{(\alpha m_{\mathrm{e}}c/n)^2}{2m_{\mathrm{e}}}$$
$$= \frac{\alpha^2 m_{\mathrm{e}}c^2}{2n^2} = -E_n = K_n. \tag{A8}$$

Here, Eq. (A4) can be derived by canceling out the term $(1 - \alpha^2/4n^2)$ in the numerator and the denominator.

Incidentally, with Bohr's model of the atom, the theory was constructed by regarding the atomic nucleus as a point. This corresponds to setting $\alpha^2/4n^2$ to zero on the right side of Eq. (A6b). If approximation is done here in this way, then Eq. (A6a) becomes

$$p_n^2 \approx \left(\frac{\alpha m_{\mathrm{e}}c}{n}\right)^2 = \frac{\alpha^2 m_{\mathrm{e}}c^2}{2n^2}\cdot 2m_{\mathrm{e}} = 2m_{\mathrm{e}}K_n. \tag{A9}$$

Thus,

$$K_n \approx \frac{p_n^2}{2m_{\mathrm{e}}}. \tag{A10}$$

The above considerations show that the correct equation is Eq. (A4).

Here, let us compare the Bohr radius Eq. (9) with the atomic nucleus of the hydrogen atom found in this paper, i.e., with the radius of the proton Eq. (54).

$$\frac{r_{\mathrm{e}}}{4}\cdot\frac{1}{r_n} = \frac{1}{4}\cdot\frac{e^2}{4\pi\varepsilon_0 m_{\mathrm{e}}c^2}\cdot\frac{m_{\mathrm{e}}e^2}{4\pi\varepsilon_0\hbar^2}\cdot\frac{1}{n^2}$$
$$= \frac{1}{4}\left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)^2\cdot\frac{1}{n^2}$$
$$= \frac{\alpha^2}{4n^2}. \tag{A11}$$

From this, the following relationship can be derived:

$$1 - \frac{\alpha^2}{4n^2} = 1 - \frac{r_{\mathrm{e}}}{4r_n} = \frac{r_n - r_{\mathrm{e}}/4}{r_n}. \tag{A12}$$

Also, taking Eq. (50) into consideration,

$$\left(\frac{p_n r_n}{n\hbar}\right)^2 = 1 - \frac{\alpha^2}{4n^2} = \frac{r_n - r_{\mathrm{e}}/4}{r_n}. \tag{A13}$$

If Eq. (35) is also taken into consideration, then we can derive the following relationship:

$$1 - \frac{\alpha^2}{4n^2} = 1 + \frac{E_n}{2E_0} = 1 + \frac{V(r_n)/4}{m_{\mathrm{e}}c^2}. \tag{A14}$$

Finally, if the new quantum condition Eq. (46) is extended, so it can be applied not only to the circular orbital but also in the case of the elliptical orbital, then it can be expressed as

$$\oint pds = 2\pi n\hbar\sqrt{\frac{r_n - r_e/4}{r_n}}. \tag{A15}$$

With the $r$ in Eq. (3) and the orbital radius of the electron $r_n$, the distance from the center of the atomic nucleus to the classical electron orbital becomes a problem. In contrast, as is clear from the $r_n - r_e/4$ in Eq. (A13), it can be predicted that the distance from the surface of the atomic nucleus to the electron orbital is involved in the angular momentum of the electron.

## ACKNOWLEDGMENTS

Logic from the textbook of French was used in deriving Eq. (33). I wish to express my gratitude to French. Also, I would like to express my thanks to the staff at ACN Translation Services for their translation assistance. Moreover, I have modified this manuscript based on the valuable comments of a reviewer. I would like to express my gratitude to the reviewer.

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